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Harmonic functions in random graphs

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Graphs in $\mathbb{R}^d$

A non-oriented graph $\mathcal{G} = (V, E)$ is a finite or countable set of vertices $V \subset \mathbb{R}^d$ and a symmetric set of *edges* $E \subset \{\{v, v'\} : v, v' \in V\}$.

v $v' \in V$ are said *neighbors* if $\{v, v'\} \in E$. We write $v \sim v'$ and $c(x, y) = \mathbf{1}\{x \sim y\}$. Let $c(x) = \sum_y c(x, y)$ be the number of neighbors of x.

Both V and E may be random.

Connected graphs.

Random walk on graphs Consider a X_t discrete time random walk on V, with transition matrix

$$Q(x, y) = \frac{c(x, y)}{c(x)}$$

The walk jumps from x to y with probability inversely proportional to the number of neighbors of x.

Reversible measure The locally finite measure μ on V defined by $\mu(x) = c(x)/Z$ (for any constant Z) is reversible:

$$\mu(x)\frac{c(x,y)}{c(x)} = \mu(y)\frac{c(x,y)}{c(y)}$$

These are the *detailed balance* equations. If μ is reversible, then it is **invariant**. Summing on x we get the *balance equations*:

$$\sum_x \mu(x)\frac{c(x,y)}{c(x)} = \mu(y).$$

If $\sum_x c(x) < \infty$ and $Z := \sum_x c(x)$, then μ is a **probability**.

If V is finite and connected, μ is unique invariant measure.

If Z is infinite, then μ is still invariant (it satisfies the balance equations) but it is not a probability.

Example: the measure $\mu(x) =$ constant is invariant for the nearest neighbor random walk in $\mathbb{Z}^d$.

Random walk games Let $V = B \cup \partial B$. Assume for each $y \in \partial B$ there is a $x \in B$ such that $\{x, y\} \in E$. Let $g : V \to \mathbb{R}$ be a "prize" function fixed on ∂B and consider the game defined on the graph (V, E) by (1) start the random walk at vertex $x \in B$. (2) perform the random walk until it hits ∂B. If it hits $y \in \partial B$, then you win $g(y)$.

Expected prize. Let $p(x, y)$ be the probability that the walk starting at x hits $y \in \partial B$ before hitting any other point in ∂B.

Then if we call $f(x)$ the expected prize, we have

$$f(x) = \sum_{y \in \partial B} p(x, y) f(y)$$

and, conditioning to the first jump of the walk:

$$f(x) = \frac{1}{c(x)} \sum_{z} c(x, z) \sum_{y \in \partial B} p(z, y) f(y) = \frac{1}{c(x)} \sum_{z} c(x, z) f(z)$$

with $f(y) = g(y)$ for $y \in \partial B$.

In other words,

$$\Delta f(x) := \frac{1}{c(x)} \sum_y c(x,y)\,[f(y) - f(x)] = 0$$

(the *Laplacian*).

We have partitioned $V = B \dot\cup \partial B$, B is the *bulk* and ∂B is the *boundary*.

Examples Nearest neighbors Random walk in a finite interval $[a,b] \subset \mathbb{Z}$. f satisfies:

$$f(x) = \frac{f(x+1) + f(x-1)}{2}, \qquad a < x < b$$

that is, must be linear with values $f(a) = g(a)$ and $f(b) = g(b)$. Hence

$$f(x) = \frac{b-x}{b-a} g(a) + \frac{x-a}{b-a} g(b)$$

Harmonic functions $f : V \to \mathbb{R}$ is *harmonic* in x if $\Delta f(x) = 0$. f above is harmonic in B and has its values fixed at ∂B.

Harmonic functions attain their extrema at the boundary:

Lemma (Maximun principle) *Consider a function* $g : V \to \mathbb{R}$ *and* $V = B \dot\cup \partial B$. *Let* f *be a harmonic function on* B *with boundary conditions* $f(x) = g(x)$ *for* $x \in \partial B$. *Then* $\max_{x \in V} f(x) \in \partial B$ *and* $\min_{x \in V} f(x) \in \partial B$.

Proof Since a harmonic function f equals the average of the neighbors, if for some $x \in B$, $f(x) = M$, the maximum, then all neighbors of x must be equal to M. Inductively, we show that all vertices, including the boundary ones, must attain the maximum. $\square$

Lemma (Uniqueness) *If* V *is finite, then there is only one harmonic function for any partition* B *and* ∂B *and any* $g : \partial B \to \mathbb{R}$.

Proof Let f, h harmonic with boundary condition g. Then $f-h$ is harmonic with boundary conditions $f(x)-h(x)\equiv 0$, for $x\in\partial B$. Since the max and the min are zero, $f-g\equiv 0$ in V. □

Symmetric random walk: f linear en $[-N,N]$

Planes in two dimensions.

Martingales Let $\mathcal{F}_n$ be a filtration. A *martingale* with respect to the filtration $\mathcal{F}_n$ is a stochastic process satisfying

$$E[M_{t+1}|\mathcal{F}_t]=M_t. \qquad (*)$$

Consider a graph (V,E) with $V\in\mathbb{R}^d$ and the translation operator $\tau_x(V,E)=(\{v-x:v\in V\},\{\{v-x,v'-x\}:,\{v,v'\}\in E\})$, the graph from the point of view of x.

Let X_n be a random walk on $G=(V,E)$ and $G_n=\tau_{X_n}G$, the graph as seen from the point of view of the random position X_n.

Let $\mathcal{F}_n=\sigma(G_1,\dots,G_n)$, the filtration induced by the process G_n.

Let $\tau = \min\{n \geq 0 : X_n \in \partial B\}$, a stopping time for $\mathcal{F}_n$. Let $\tilde{X}_t := X_{\tau \wedge t}$ be the process stopped at ∂B: when it reaches a state in ∂B, it stops.

Lemma *If h is harmonic in B, then $M_t := h(\tilde{X}_t)$ is a martingale.*

Proof If $x \in \partial B$ and $X_t = x$, then $X_{t+1} = x$ and (*) holds.

$\mathcal{F}_t$ is the filtration generated by $X_0, \ldots, X_t$. If $x \in B$,

$$E[M_{t+1}|X_t = x] = \frac{1}{c(x)} \sum_y c(x,y)h(y) = h(x)$$

because h is harmonic in x. Hence, $E[M_{t+1}|\mathcal{F}_t] = M_t$. $\square$

If we start with a fortune $h(x)$, then $h(X_t)$ is the (random) expected fortune after t steps and $Eh(X_t) = h(x)$. So that the expected fortune at any time is the same as the initial fortune.

If you start with a fortune $h(x)$ at x, then the expected final fortune is

$$E(M_\tau | X_0 = x) = \sum_{y \in \partial B} h(y) P(X_\tau = y | X_0 = x) = h(x) = EM_0$$

Harmonic functions minimize elestrostatic energy

For a function $f : V \to \mathbb{R}$ let

$$\mathcal{E}(f) = \frac{1}{2} \sum_{x,y \in V} c(x,y)(f(x) - f(y))^2$$

Fix subgraph $B \subset V$ and a boundary condition function $g : \partial B \to \mathbb{R}$.

Let $M_x : \mathbb{R}^B \to \mathbb{R}^B$ be the operator defined by

$$M_x f(y) := \begin{cases} f(y) & y \neq x \\ \sum_y \frac{c(x,y)}{c(x)} f(y) & y = x. \end{cases}$$

Lemma *If $f : V \to \mathbb{R}$ such that $f(x) = g(x)$ for $x \in \partial B$, then $\mathcal{E}(M_y f) \leq \mathcal{E}(f)$ for all $y \in B$.*

Proof This is an exercise of descriptive statistics: the expectation of a discrete random variable minimizes the weighted sum of the squares. □

Lemma *Let $f : V \to \mathbb{R}$ be a function such that $f(x) = g(x)$ for $a \in \partial B$. Let y_n be a sequence of vertices in B containing infinitely many times each vertex. Let $f_0 = f$ and for $n \geq 1$, $f_n = M_{y_n} f_{n-1}$. Then $\lim_n f_n = h$, the unique harmonic function on B with boundary conditions g.*

Proof Assume first that $\min g \leq f \leq \max g$. Take $f = \min g$. By the previous lemma, $\mathcal{E}(f_n)$ is non increasing and bounded by $\mathcal{E}(f)$, hence $\mathcal{E}(f_n)$ converges and $f_n(x)$ is uniformly bounded in n and $x \in V$. On the other hand, $f_n \leq f_{n+1}$, and $f_n \leq \max g$. Let $f_\infty = \lim_n f_n$. Let $\mathcal{E}(f_\infty)$. Finally $M_x f_\infty = f_\infty$ for all $x \in B$ because the sequence visits infinitely often the site x. Hence, f_∞ is harmonic. The same argument shows that if $f = \max g$, then

$\lim_n f_n$ is harmonic. By uniqueness of the harmonic function, both limits agree. By monotonicity, any function between the min and the max of g must converge to the harmonic function.

Since there is a unique harmonic function, the lemma follows. □

Corollary *Let $f : V \to \mathbb{R}$ such that $f(x) = g(x)$ for $x \in \partial B$. Then $\mathcal{E}(f) \geq \mathcal{E}(h)$, where h is the unique harmonic function in B with boundary conditions g.*

Proof Immediate. □

Proposition *f is harmonic in B with boundary conditions g in ∂B if and only if f is a minimizer of the set*

$$\{\mathcal{E}(f, B, g) : f : V \to \mathbb{R},\ f(x) = g(x),\ x \in \partial B\} \tag{1}$$

Proof In a harmonic function the height at x is the mean of the heights at the neighbors of x. The mean minimizes the sum of the squares of the differences of the heights. Hence if f is a minimizer, then each height in B must be the mean of the neighbors (otherwise I could move the height at one site and get a strictly smaller height). Reciprocally, let g_n be a sequence of functions such that $\mathcal{E}(g_n) \searrow$ the infimun of the set (1). Since by the previous lemma $\mathcal{E}(g_n) \geq \mathcal{E}(h)$ for all n, the infimun dominates $\mathcal{E}(h)$. Hence h is the unique minimizer. $\square$

Construction of harmonic functions in finite graphs

Problem: Consider the finite graph (V, E) with $V = B \dot{\cup} \partial B$ and a function $a : \partial B \to \mathbb{R}$ and construct $H : V \to \mathbb{R}$ harmonic in B and with boundary conditions $H(x) = g(x)$ for $x \in \partial B$.

Since the graph is finite, we know there is a unique solution H and that this solution minimizes que electrostatic energy $\mathcal{E}$:

$$\mathcal{E}(H) \leq \mathcal{E}(f), \text{ for all } f : V \to \mathbb{R} \text{ with } f(x) = H(x) \text{ for } x \in \partial B.$$

Solution to a linear problem We have a system of linear equations

$$H(x) = \frac{1}{c(x)} \sum_{y \in V} c(x,y) H(y), \quad x \in B.$$

with boundary conditions $H(y)$ for $y \in \partial B$. In matrix form, this is the solution of

$$Q_B H = U$$

where Q_B is random walk matrix Q_B given by $Q(x,y) = \frac{c(x,y)}{c(x)}$, $x, y \in B$ and

$$U(x) = -\sum_{y \in \partial B} \frac{c(x,y)}{c(x)} H(y), \quad x \in B.$$

So that, if the matrix Q_B is invertible,

$$H = Q_B^{-1} U.$$

Harness process η_t on $\mathbb{R}^B$. Fix the values of $\eta_t(x) = H(x)$ for any $x \in \partial B$ and starting with a surface η_0, at rate 1 the height of the surface at vertex $x \in B$ is updated to the mean value of its neighbors.

So that, we have a continuous time jump Markov process on $\mathbb{R}^V$ whose generator is

$$\mathcal{L}f(\eta) = \sum_{x \in B} \mathcal{L}_x f(\eta)$$

with

$$\mathcal{L}_x f(\eta) = f(M_x \eta) - f(\eta)$$

Since the mean minimizes the sum of the squares, updating vertex x strictly decreases $\mathcal{E}(\eta_t)$, unless η_t is harmonic in x. More precisely

$$L_x \mathcal{E}(\eta) \leq 0, \quad x \in V$$

and the inequality is strict unless η is harmonic in x.

Lemma η_t *converges almost surely to h starting from any configuration* η_0.

Proof In the graphical construction of the process apply the previous lemma to the sequence y_n of updating sites. This holds because all sites will be updated infinitely often for almost all realization of the Poisson processes. (Finite volume). $\square$

Harmonic functions in infinite graphs

Consider the infinite graph $G = (\mathbb{Z}^2, \mathbb{E}^2)$, with $\mathbb{E}^2 = \{\{x, y\} \subset \mathbb{Z}^2 : |x - y| = 1\}$, the edges induced by nearest neighbors.

No boundary conditions A *harmonic function* in G is function $f : V \to \mathbb{R}$ satisfying

$$\Delta f(x) = 0, \quad \text{for all } x \in V$$

Let the "energy" be defined by

$$\mathcal{E}(f) = \frac{1}{2} \sum_{x,y \in V} c(x, y)(f(x) - f(y))^2$$

not well defined in infinite volume.

Harmonic functions "minimize energy" as follows:

Let h harmonic and $g \equiv h$ in $V \setminus V'$, a finite subset of vertices.

Then $\mathcal{E}(g) - \mathcal{E}(h) > 0$ if $g \neq h$.

A *hyperplane* is a function $f : \mathbb{Z}^2 \to \mathbb{R}$ given by $f(x) = \langle x, y \rangle + c$, where $\langle \cdot, \cdot \rangle$ is inner product, y is an arbitrary vector in $\mathbb{R}^2$ and c is a constant.

Hyperplanes in $\mathbb{Z}^2$ are harmonic functions.

The trivial harmonic function is $h(x) \equiv 0$.

Saddle harmonic function $x = (x_1, x_2) \in \mathbb{Z}^2$. The function $h(x_1, x_2) = x_1^2 - x_2^2$ is harmonic.

Liouville theorem: if $|h(x) - h(y)|c(x, y) < \infty$, for all $x, y \in \mathbb{Z}^d$, then h is linear.

With boundary conditions Can also partition $V = B \cup \partial B$ and g boundary condition and ask for h harmonic in B with $h = g$ in ∂B. If the random walk is transient, then one needs to add a boundary condition at infinity. Let τ be the hitting time of ∂B and c be the "condition at infinity". Then

$$h(x) = Eg(X_\tau^x)$$

with the convention $g(X_\infty^x) = c$. For instance, $g(a) = 0$, $g(b) = 1$ for arbitrary points in $\mathbb{Z}^d$.

Discrete harness process Update each vertex at each time to the average of the neighbors simultaneously: Fix $\eta(\cdot, 0) : V \to \mathbb{R}$ and define for $t \geq 0$,

$$\eta(x, t+1) = \sum_y \frac{c(x,y)}{c(x)} \eta(y, t) \tag{2}$$

in "pde" form:

$$\eta(x, t+1) - \eta(x, t) = \Delta\eta(x, t), \qquad x \in \mathbb{Z}^d,\ t \in \mathbb{N}$$

with initial condition $\eta(\cdot, 0)$

Harmonic functions are invariant for η_t. $\eta(\cdot, t) = h$ if $\eta(\cdot, 0) = h$.

There are also "travelling harmonic functions":

Quadratic: $\eta(x, t) = ax^2 + 2at + b$ is a solution of (3).

Exponential: $H(x) = ae^{bx} + bt$ is a solution of (3).

Random graphs

Consider a translation invariant point process $S \subset \mathbb{R}^d$ of rate 1 and a graph having S as vertices. The edges may be random or a deterministic function of S.

Palm version: S^o the process S conditioned to have a point at the origin.

Point translation invariant. $0 \in S^o$ and for any bounded $\Lambda \subset \mathbb{Z}^d$,

$$Ef(S^o) = \frac{1}{\ell(\Lambda)} \sum_{s \in \Lambda} Ef(\tau_s S)$$

Use notation S instead of S^o.

Examples:

1. Infinite cluster of **supercritical** (bond or site) **percolation** in Z^d.

Take the cluster and eliminate the dead ends: iteratively eliminate vertices having only one neighbor.

2. Point process with **random energy edges**. At each pair of sites $s, s' \in S$, include the edge $\{s, s'\}$ with probability $\exp(-\beta\|s - s'\|)$.

For β sufficiently small there is an infinite cluster. Eliminate dead ends and consider the graph induced by the infinite cluster.

3. **Delaunay triangulation** of a translation invariant point process. We will work in this case with S Poisson(1).

Poisson process in $\mathbb{R}^d$. $S \cup \{0\}$ is the Palm version of a homogeneous Poisson process on $\mathbb{R}^d$ (with a point at the origin).

$\mathbb{P}$, $\mathbb{E}$ probability and expectation on $\mathcal{N}$ induced by S.

points $s \in S$ and *sites* $x \in \mathbb{R}^d$

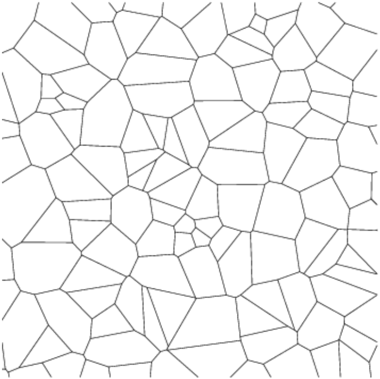
Figure: Poisson process.

Voronoi tessellation. Voronoi cell

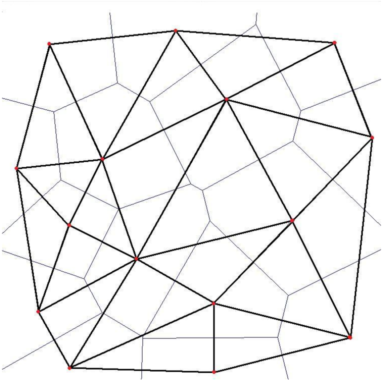
$$\mathrm{Vor}(s) = \{x \in \mathbb{R}^d : |x - s| \leq |x - s'|, \text{ for all } s' \in S \setminus \{s\}\}$$

Voronoi neighbors share a $(d-1)$-dimensional boundary.

$c(s, s') := \mathbf{1}\{s \text{ and } s' \text{ are neighbors}\}$



Delaunay triangulation is the *graph* $\mathcal{G} = (S, \mathcal{E})$ with $\mathcal{E} := \{(s, s') : s \text{ and } s' \text{ are neighbors}\}$.



Surfaces are functions $\eta : \Xi_1 \to \mathbb{R}$, where

$$\Xi_1 := \{(s, S) \in \mathbb{R}^d \times \mathcal{N} : s \in S\}.$$

We look for harmonic surfaces, that is, satisfying for all $s \in S$

$$\Delta h(s, S) = 0$$

The surface $h(s, S) \equiv 0$ is trivially harmonic.

Thermodynamical limit with plane boundary conditions.

$y \in \mathbb{R}^d$, $c \in \mathbb{R}$. Let $\pi \in \Xi$ "plane" $\pi(s,S) := \langle s, y\rangle + c$.

Look for a harmonic function "associated" to each plane π.

Let $\Lambda \subset \mathbb{R}^d$ be a bounded set.

Let $h_\Lambda(\cdot, S) : S \to \mathbb{R}$ be the unique harmonic surface in $S \cap \Lambda$ with boundary conditions

$$h_\Lambda(s,S) := \pi(s,S), \quad s \in S \cap \Lambda^c$$

Explicit solution in finite volume:

$$h_\Lambda(s) = \sum_{y \in S \cap \Lambda^c} \rho(s,y)\pi(y)$$

where $\rho(s,y)$ is the probability that a random walk on $(S,\mathcal{E})$ starting at s hits y before any other point in $S \cap \Lambda^c$.

Problem Is there a limit as $\Lambda \nearrow \mathbb{Z}^d$ of h_Λ?

$d = 1$, take $\pi(s, S) = s$ for all $s \in S$.

Write h_N instead of $h_{[-N,N]}$ and attempt to compute $\lim_N h_N$.

Label points: $S = \{\dots, s_{-1}, 0, s_1, s_2, \dots\}$ $s_0 = 0$, $s_i < s_{i+1}$

$$h_N(s_k) = s_{R_N} \frac{k - L_N}{R_N - L_N} + s_{L_N} \frac{R_N - k}{R_N - L_N}$$

L_N is the label of the first point to the left of $-N$.

R_N is the label of the first point to the right of N.

$s_{L_N} = -N-$ exp(1), $s_{R_N} = N+$ exp(1)

$$h_N(0) = \frac{N}{R_N - L_N}(-L_N - R_N + O(1)) = \frac{-L_N - R_N}{2}$$

because $(R_N - L_N)/N$ converges to 2 in probability

$$h_N(0) = \frac{-L_N - R_N}{2} \sim N(0, N/2)$$

(difference between to independent Poisson of mean N)

Hence $|h_N(0)| \sim \sqrt{N}$. So that $h_N(0)$ delocalizes as $N \to \infty$.

Gradients: Suppose now that s_i and s_{i+1} are successive points. Hence

$$h_N(s_{i+1}) - h_N(s_i) = 1$$

In this case, if s_k is the k-th point of S, then the limiting harmonic function as seen from the origin is

$$h(s_k) - h(0) = k$$

Let $\pi(s) = s$ be the line with inclination 1.

Let $\mathrm{Cen}(k) = \arg\min\{|s-k| : s \in S\}$, the center of the Voronoi cell containing k.

$$\frac{[h_N(\mathrm{Cen}(k)) - h_N(0)] - \pi(k)}{k} = \frac{S[0,k] - k}{k} \sim N\Big(0, \frac{1}{k}\Big).$$

So that $h(\mathrm{Cen}(k)) - k$ is of order $\sqrt{k}$. We say that h is a sub-linear perturbation of the linear function $\pi(k) \equiv k$.

$h(s) - \pi(s)$ is called the *corrector*.

Surface Inclination.

$u \in \mathbb{R}^d$ unit vector. Surface η has *inclination* $\mathcal{I}_u(\eta)$ in the direction u if the following limit exists and does not depend on s

$$\mathcal{I}_u(\eta) := \lim_{K\to\infty} \frac{\eta(\mathrm{Cen}(s+Ku)) - \eta(s)}{K} \qquad \mathbb{P}\text{-a.s.} \tag{3}$$

where $\mathrm{Cen}(x)$ is the point in S closest to $x \in \mathbb{R}^d$.

Harness process.

Let $M_s\eta$ the surface obtained by substituting the value of $\eta(s)$ with the average of the heights at the neighbors of s:

$$(M_s\eta)(v) = \begin{cases} \frac{1}{c(s)} \sum_{s' \in S \setminus \{s\}} c(s,s')\, \eta(s'), & \text{if } v = s \\ \eta(v), & \text{if } v \neq s \end{cases}$$

The harness process η_t is the Markov process with generator

$$Lf(\eta) = \sum_{s \in S} [f(M_s\eta) - f(\eta)]$$

At rate 1, the height at s is updated to the average of the heigths at the neighbors of s.

Construction of the harness process

Enumerate the points of S in a point-translation invariant way (Holroyd-Peres).

Associate to each point $s \in S$ a (time) one-dimensional Poisson process or rate 1.

These processes are independent.

Use these times to update the corresponding site.

Use the same notation $\mathbb{P}$ and $\mathbb{E}$ for the product of the law of S and the time Poisson processes.

More definitions

Fields are functions $\zeta : \Xi_2 \to \mathbb{R}$ where

$$\Xi_2 = \{(s, s', S) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{N} : s, s' \in S\}.$$

Will drop dependence on S.

Gradient of a surface η is the field $\nabla\eta$ defined by

$$\nabla\eta(s, s') \;=\; (\eta(s') - \eta(s))$$

Let $\tau_s S = \{s' - s,\ s' \in S\}$.

A field $\zeta : \Xi_2 \to \mathbb{R}$ is **covariant** if

$$\zeta(s' - s, s'' - s, \tau_s S) \;=\; \zeta(s', s'', S)$$

Theorem [F., Grisi, Groisman]

(a) if $\eta_0(s) = s_1$ where s_1 is the first coordinate of s, then $\eta_t(\cdot) - \eta_t(0)$ converges in L_2 to a surface $h : \Xi_1 \to \mathbb{R}$:

$$\lim_{t\to\infty} \mathbb{E}[(\eta_t(s) - \eta_t(0)) - h(s)]^2 \;=\; 0$$

$\mathbb{E}$ is with respect to the product measure of the spacial and temporal Poisson processes.

(b) The limit h is harmonic, has covariant gradient and inclination 1 in the direction e_1, $\mathbb{P}$-a.s..

Percolation: Berger-Biskup, Mathieu-Piatnitski;

Poisson + energy marks: Caputo-Faggionato-Prescott.

The space of fields as a Hilbert space

$$\Xi_2 = \{(s, s', S) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{N} : s, s' \in S\}.$$

S is Palm of a Poisson process in $\mathbb{R}^d$ with law $\mathbb{P}$, $\mathbb{E}$.

For a field $\zeta : \Xi_2 \to \mathbb{R}$ define

$$\mathcal{C}(\zeta) = \mathbb{E}\left[\sum_{s \in S} c(0, s)\zeta(0, s)\right].$$

Hilbert space $\mathcal{H} := L^2(\Xi_2, \mathbb{R}, \mathcal{C})$. **Inner product:** for fields ζ, ζ' in $\mathcal{H}$:

$$\mathcal{C}(\zeta \cdot \zeta') = \mathbb{E}\left[\sum_{s \in S} c(0, s)\zeta(0, s)\zeta'(0, s)\right].$$

Cesàro limit of covariant fields

Let $\zeta \in \mathcal{H}$ be a covariant field and define

$$C(\zeta) := \lim_{\Lambda \nearrow \mathbb{R}^d} \frac{1}{2|\Lambda|} \sum_{s \in S \cap \Lambda,\, s' \in S} c(s, s')\zeta(s, s').$$

Since S is ergodic, by the Point Ergodic Theorem we have that almost surely

$$C(\zeta) = \mathcal{C}(\zeta)$$

and

$$C(\zeta, \zeta') = \mathcal{C}(\zeta, \zeta')$$

Inclination as inner product

Fix unitary vector $e \in \mathbb{R}^d$

For Voronoi neighbors $v, w \in S$ define:

$p(v,w)$:= $(d-1)$-dimensional common side of cells of v and w

$p_e(v,w)$:= $(d-1)$-dimensional Lebesgue measure of the projection of $p(v,w)$ over the hiperplane perpendicular to e.

v_e := $e\langle v, e\rangle$ (projection of v over the line determined by e).

Define the field κ_e by

$$\kappa_e(v,w) := \frac{1}{2}\,\mathrm{sg}(v_e - w_e)\, p_e(v,w)\, c(v,w)$$

Remark: $\kappa_v \in \mathcal{H}$ is covariant.

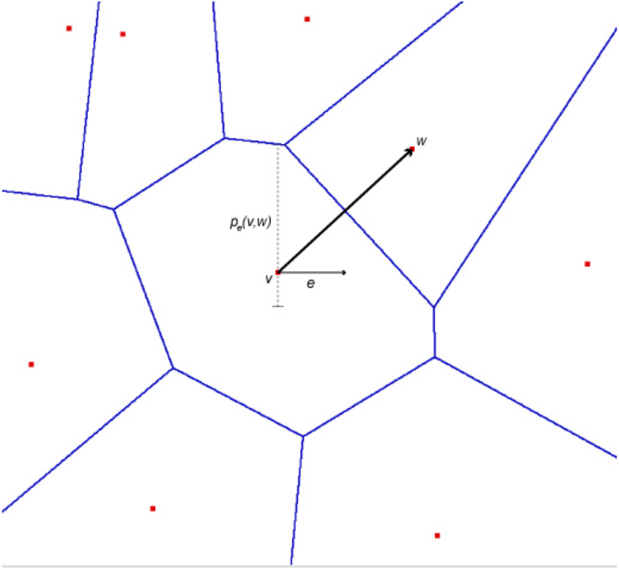


Figure: Definition of the field κ_e with $e = (1, 0)$.

From the definition:

$$\sum_{s' \in S} \kappa_e(0, s') \;=\; \frac{1}{2} \sum_{s'} c(0, s') \, \mathrm{sg}(s'_e) \, p_e(0, s') \;=\; 0$$

the projections of the "negative" sides has the same area as the projections of the "positive" sides.

By covariance, for all $s \in S$:

$$\sum_{s' \in S} \kappa_e(s, s') \;=\; 0$$

Anti-symmetry:

$$\kappa_e(s, w) = -\kappa_e(w, s)$$

Also gradient is anti-symmetric.

Second definition of inclination:

$$\mathcal{J}_e(\eta) \;:=\; \mathcal{C}(\nabla\eta \cdot \kappa_e)$$

$$\begin{aligned}
\mathcal{J}_e(\eta) &= \frac{1}{2}\,\mathbb{E}\sum_{s\in S} c(0,s)\,(\eta(s)-\eta(0))\,\kappa_e(0,s) \\
&= \lim_{\Lambda\nearrow\mathbb{R}^d} \frac{1}{2|\Lambda|} \sum_{s\in S\cap\Lambda,\, s'\in S} c(s,s')\,(\eta(s')-\eta(s))\,\kappa_e(s,s').
\end{aligned}$$

Proposition *Let η be a surface with covariant $\nabla\eta \in \mathcal{H}$. Then*

$$\mathcal{I}_e(\eta, S^o) = \mathcal{J}_e(\eta) \qquad \mathbb{P}\text{-a.s.}$$

Proof.

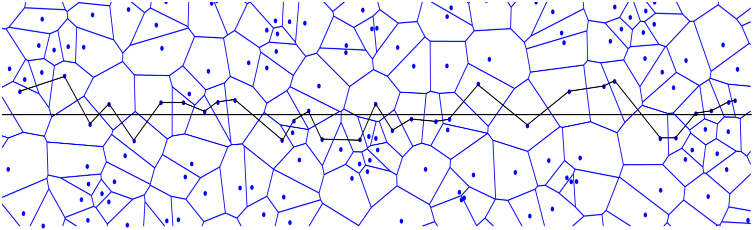


Figure: Points contributing to the inclination along the line $y = 0$.

Inclination is invariant for the dynamics:

$$\mathcal{J}_e(\eta_t) = \mathcal{J}_e(\eta_0)$$

Why? Updating the origin gives zero contribution: Let $\tilde{\eta} = M_0 \eta$

$$\begin{aligned}
\mathcal{J}_e(\eta) - \mathcal{J}_e(\tilde{\eta}) &= \mathbb{E} \sum_{s'} \kappa_e(0, s')[\nabla \eta(0, s') - \nabla \tilde{\eta}(0, s')] \\
&= \mathbb{E}\Big[(\tilde{\eta}(0) - \eta(0)) \sum_{s'} c(0, s') \, \mathrm{sg}(s'_e) \, p_e(0, s')\Big] = 0
\end{aligned}$$

because the $(d-1)$-Lebesgue measure of the projections with negative contribution coincides with the one of the projections with positive contribution.

The contributions of the updating of neighbors of the origin are also zero by translation invariance and covariance of the fields involved.

Sketch of proof of the Theorem

(a) Convergence of the gradients of the harness process starting with an hyperplane to a limit in $L_2(\mathcal{C})$.

(b) Limiting field is the gradient of a harmonic surface with inclination 1

To show (a) we show:

(1) the gradients of the harness process starting with a hyperplane converge to a field

(2) the limit field is the gradient of a harmonic surface with inclination 1. 1 in direction e_1.

Recall η_t is the harness process with $\eta_0 =$ "hyperplane".

Ingredients:

1) Integration by parts formula:

$\zeta \colon \Xi_2 \to \mathbb{R}$ covariant field

$\psi \colon \Xi_1 \to \mathbb{R}$ translation invariant surface ($\psi(v, S) = \psi(0, \tau_v S)$)

such that $\nabla\psi, \zeta \in \mathcal{H}$. Then

$$\mathcal{C}(\nabla\psi \cdot \zeta) \; = \; -\mathbb{E}[\psi(0)\,\mathrm{div}\zeta(0)]$$

where the *divergence* is given by

$$\mathrm{div}\zeta(s) = \sum_{s' \in S} \zeta(s, s')$$

Used for $\psi_t = \eta_t - \eta_0$.

Write $\eta_t = \eta_0 + \psi_t$, where η_0 is a “hyperplane” and ψ_t is translation invariant.

2) Square of gradients decrease: For all $t > 0$

$$\frac{d}{dt}\mathcal{C}(|\nabla \eta_t|^2) \; = \; -2\mathbb{E}\left[\frac{|\Delta \eta_t(0)|^2}{c(0)}\right],$$

3) Laplacian converges almost surely to 0

$$\infty \; > \; \mathcal{C}(|\nabla \eta_0|^2) \; \geq \; \lim_{t\to\infty} \mathcal{C}(|\nabla \eta_t|^2) \; = \; 2\int_0^\infty \mathbb{E}\left[\frac{|\Delta \eta_t(0)|^2}{c(0)}\right] dt,$$

4) Weak convergence of $\nabla \eta_t$ to ζ_∞ by subsequences:

By (2) there exists a subsequence $\{t_k\}$ and a field $\zeta_\infty \in \mathcal{H}$ such that

$$\lim_{k\to\infty} \mathcal{C}(\nabla \eta_{t_k} \cdot \zeta) \; = \; \mathcal{C}(\zeta_\infty \cdot \zeta),$$

for all $\zeta \in \mathcal{H}$.

5) Uniqueness of the limit:

Using integration by parts,

$$\mathcal{C}(\nabla\eta_0 \cdot \zeta_\infty) = \mathcal{C}(|\zeta_\infty|^2).$$

$$\mathcal{C}(\tilde{\zeta}_\infty \cdot \zeta_\infty) = \mathcal{C}(|\zeta_\infty|^2) = \mathcal{C}(|\tilde{\zeta}_\infty|^2).$$

6) Convergence in L_2 and a.s. along subsequences.

Using Holder and convergence of Laplacian to zero,

$$\lim_{t\to\infty} \mathcal{C}(|\nabla\eta_t - \zeta_\infty|^2) = 0.$$

7) Limit ζ_∞ is covariant.

Follows from the covariance of $\nabla\eta_t$ for each t and a.s. convergence along subsequences.

8) The limiting field has zero divergence. Hölder:

$$\lim_{t\to\infty} \mathbb{E}(c(0)^{-2}|\Delta\eta_t - \mathrm{div}\zeta_\infty|^2) \le \lim_{t\to\infty} \mathcal{C}(|\nabla\eta_t - \zeta_\infty|^2) = 0.$$

implies

$$\mathrm{div}(\zeta_\infty) = 0 \qquad \text{a.s.}$$

9) The limit is a gradient field

Convergence in L_2 implies there exists a subsequence converging almost surely. This sequence must satisfy the cocycle property.

10) The limit is the gradient of a harmonic surface

Follows from (8) and (9).

11) The limit has the same inclination as η_0

This is because the inclination $\mathcal{J}$ is invariant for the dynamics:

$$\mathcal{J}_e(\eta_t) = \mathcal{C}(\nabla\eta_0 \cdot \kappa_e) = \mathcal{C}(\nabla\eta_t \cdot \kappa_e) \longrightarrow \mathcal{C}(\nabla\eta_\infty \cdot \kappa_e)$$

Generalization

Theorem holds if S is the Palm version of a stationary point process in $\mathbb{R}^d$ and

- **A1** The law of S is mixing. (To get one-dimensional LLN)
- **A2** For every ball $B \subset R^d$, $|S \cap \partial B| < d + 2$.
- **A3** $\mathbb{E}\exp(\beta c(0, S)) < \infty$ for some positive constant β.
- **A4** $\mathcal{C}(\omega_u^2) < \infty$ for every $u \in \mathbb{R}^d$.
- **A5** S aperiodic, meaning that $\mathbb{P}(\exists x \in \mathbb{R}^d \setminus \{0\} : \tau_x S = S) = 0$.
- **A6** $\mathbb{E}[\sum_{s \in S} c(0, s)|s|^r] < \infty$ for some $r > 4$.
- **A7** $S = \{s_n;\ n \in \mathbb{Z}\}$, and $\tau_{s_n} S \overset{law}{=} S$ for every $n \in \mathbb{Z}$.
- **A8** $\mathbb{E}[\ell(\mathrm{Vor}(0, S))^2] < \infty$.

Open problems

Uniqueness of the harmonic function with given inclination.

Uniform sublinearity of the corrector.

Thermodynamic limit.

Almost sure convergence of $\nabla\eta_t$ to ∇h

Other Hamiltonians.

Harmonic graphs

Coordinates of a harmonic graph are harmonic surfaces:

Let $H : S \to \mathbb{R}^d$ and $h_1(s), \dots, h_d(s)$ the coordinates of $H(s)$.

Graph H is harmonic <u>iff</u> $h_1, \dots, h_d$ are harmonic surfaces.

Let $H = (h_1, \dots, h_d)$. Then

Graph $(H(S), \widetilde{\mathcal{E}})$ is a **sub-linear** perturbation of S
<u>iff</u> coordinate h_i has **inclination 1** in the direction e_i for all i.

Harmonic sub-linear deformation

A graph with vertices in $\mathbb{R}^d$ is **harmonic** if
each point is located in the baricenter of its neighbors.

Goal: move the points of S such that keeping the Delaunay neighborhood, the resulting graph is harmonic. $H : S \to \mathbb{R}^d$ s.t.

(1) $H(S)$ is harmonic:

$$H(s) = \frac{1}{c(s)} \sum_{s' \in S} c(s,s')\, H(s'), \qquad \text{for all } s \in S.$$

where $c(s) = \sum_{s' \in S} c(s,s')$ is the number of neighbors of s.

(2) $H(S)$ is a sublinear deformation of S:

$$\lim_{K \to \infty} \frac{|H(\mathrm{Cen}(Ku)) - \mathrm{Cen}(Ku)|}{|K|} = 0, \qquad u \text{ unit vector.}$$

Corrector. $H(s) - s$ is called *corrector*.

Figure: Poisson process.

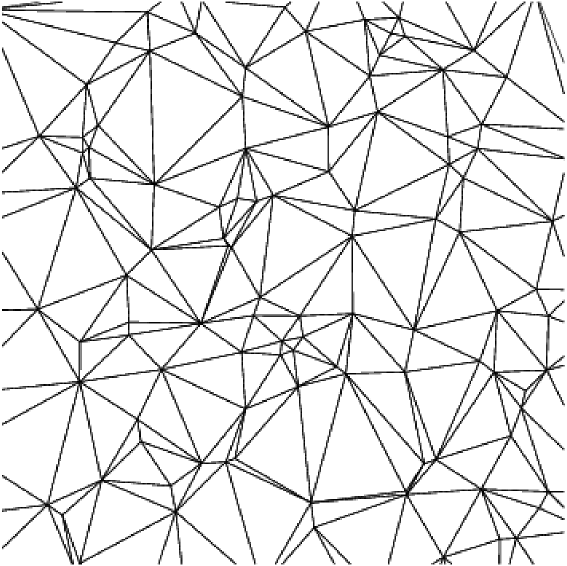


Figure: Delaunay triangulation of the above Poisson process.

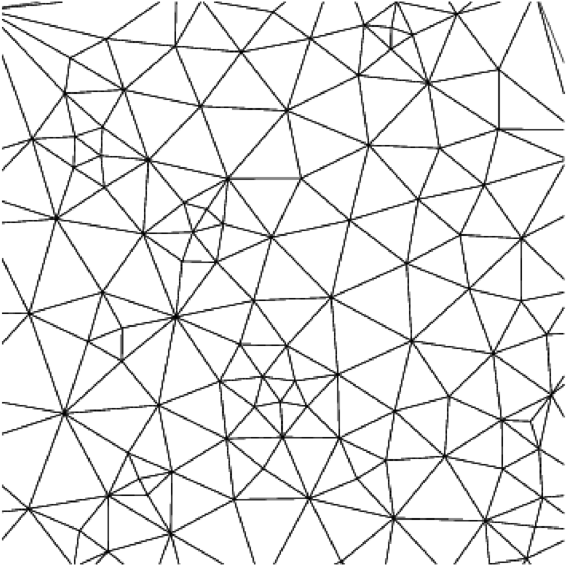


Figure: Harmonic deformation of above Delaunay triangulation.

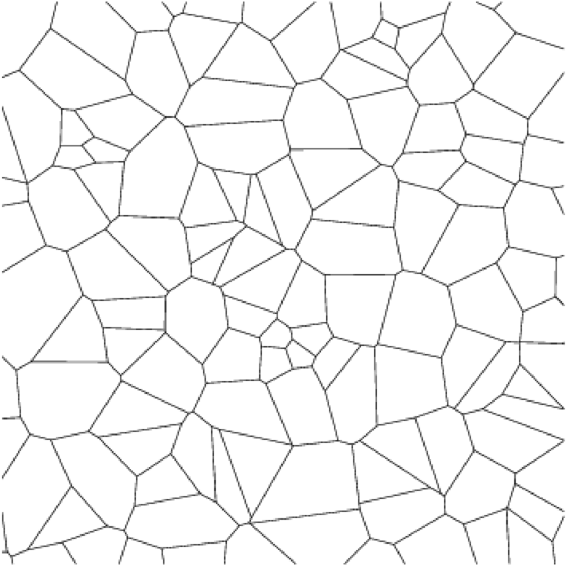


Figure: Voronoi cells of the Poisson process.

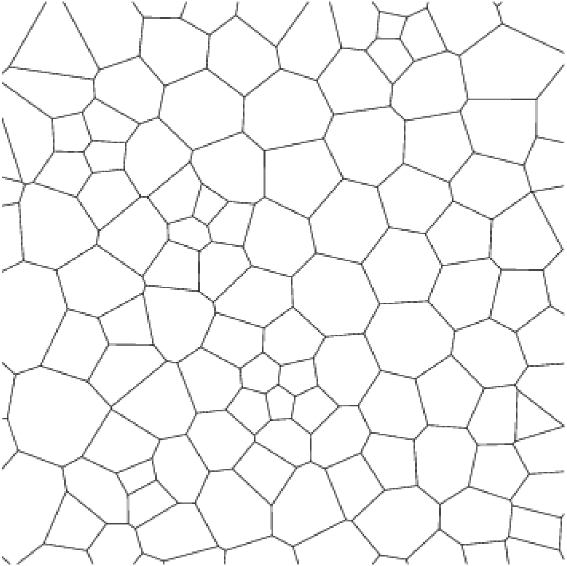


Figure: Voronoi cells of the harmonic graph.

Background

Harmonic deformation graph constructed in two settings:

Percolation clusters:

Berger and Biskup (2007) (our motivation)

Mathieu and Piatnitski (2007)

Barlow and Deuschel (2010)

Poisson process with energy marks:

Caputo, Faggionato and Prescott (2010)

Both approaches use static methods.

Application to random walk in Delaunay triangulation.

Y_t^S: random walk in the Delaunay triangulation with generator

$$L_S f(s) = \sum_{s' \in S} c(s,s')[f(s') - f(s)]$$

Since the graph H is harmonic, $H(Y_t^S)$ is a martingale and so satisfies the invariance principle $\mathbb{P}$-a.s..

To show the invariance principle for Y_t^S it suffices uniform sub-linearity of the corrector $H(s) - s$ (but seems too much).

(OK in $d = 2$ à la BB, or Heat Kernel Estimates à la Barlow)

Positive diffusion easy.

Berger-Biskup, Mathieu-Piatnitski, Sidoravicius-Snitzman, Caputo-Faggionato-Prescott and many others.

Most of the material about finite graphs is taken from the beautiful book by Doyle Snell [5].

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7. P. A. Ferrari, R. M. Grisi, P. Groisman Harmonic deformation of Delaunay triangulations. *Stoch. Process. Appl.* **6** 2185-2210, 2012, arXiv/1012.1677.

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