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Fock space representation and perturbation analysis of Poisson functionals

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1. Motivation of perturbation analysis

Setting

Φ is a Poisson process on some measurable space $(\mathbb{X}, \mathcal{X})$ with intensity measure λ. This process can be interpreted as a random element in the space **N** of all integer-valued σ-finite measures on $\mathbb{X}$, equipped with the usual (product) σ-field. By definition Φ has the following two properties:

- The random variables $\Phi(B_1), \dots, \Phi(B_m)$ are stochastically independent whenever $B_1, \dots, B_m$ are measurable and pairwise disjoint.
- $$\mathbb{P}(\Phi(B) = k) = \frac{\lambda(B)^k}{k!} \exp[-\lambda(B)], \quad k \in \mathbb{N}_0, B \in \mathcal{X},$$

  where $\infty^k e^{-\infty} := 0$ for all $k \in \mathbb{N}_0$.

Objective of perturbation analysis

- Let Φ be a Poisson process whose intensity measure λ_θ depends on a parameter $\theta \in \mathbb{R}$. For instance we might have $\lambda_\theta = \theta\lambda^d \otimes \mathbb{Q}$, where $\theta > 0$, λ^d is Lebesgue measure and $\mathbb{Q}$ is a probability measure on the (mark) space $\mathbb{Y}$.
- Consider a function $f(\Phi)$ of (the sample path) Φ and derive a formula for the derivative

$$\frac{d}{d\theta}\mathbb{E}_{\lambda_\theta} f(\Phi).$$

- Use Monte-Carlo integration to estimate the derivative.

2. Fock space representation

Definition

For $n \in \mathbb{N}$ let $\mathbf{H}_n$ be the space of symmetric functions in $L^2(\lambda^n)$, and let $\mathbf{H}_0 := \mathbb{R}$. The Fock space $\mathbf{H}$ associated with Φ (or λ) is the direct sum of the spaces $\mathbf{H}_n$, $n \geq 0$, equipped with the scalar product

$$\langle f, g \rangle_{\mathbf{H}} := \sum_{n=0}^{\infty} \frac{1}{n!} \langle f_n, g_n \rangle_n, \quad f = (f_n), g = (g_n) \in \mathbf{H},$$

where $\langle \cdot, \cdot \rangle_n$ is the scalar product in $L^2(\lambda^n)$. This is a Hilbert space.

Definition (Difference operator)

For a measurable function $f : \mathbf{N} \to \mathbb{R}$ and $x \in \mathbb{X}$ we define a function $D_x f : \mathbf{N} \to \mathbb{R}$ by

$$D_x f(\mu) := f(\mu + \delta_x) - f(\mu).$$

For $x_1, \dots, x_n \in \mathbb{X}$ we define $D^n_{x_1,\dots,x_n} f : \mathbf{N} \to \mathbb{R}$ inductively by

$$D^n_{x_1,\dots,x_n} f := D^1_{x_1} D^{n-1}_{x_2,\dots,x_n} f,$$

where $D^1 := D$ and $D^0 f = f$.

Lemma

For any $f \in L^2(\mathbb{P}_\Phi)$

$$T_n f(x_1, \dots, x_n) := \mathbb{E} D^n_{x_1,\dots,x_n} f(\Phi),$$

defines a function $T_n f \in \mathbf{H}_n$.

Theorem (L. and Penrose '11)

Let $f, g \in L^2(\mathbb{P}_\Phi)$. *Then*

$$\mathbb{E}f(\Phi)g(\Phi) = (\mathbb{E}f(\Phi))(\mathbb{E}g(\Phi)) + \sum_{n=1}^{\infty} \frac{1}{n!}\langle T_n f, T_n g\rangle_n.$$

That is,

$$\mathbb{E}f(\Phi)g(\Phi) = \langle Tf, Tg\rangle_{\mathbf{H}},$$

where $Tf := (T_n f)$ *and* $Tg := (T_n g)$.

Sketch of proof: Consider functions of the special form

$$f(\mu) := \exp[-\mu(v)], \quad g(\mu) = \exp[-\mu(w)]$$

where $v, w : \mathbb{X} \to \mathbb{R}_+$ vanishes outside a set in the system $\mathcal{X}_0$ of all measurable $B \in \mathcal{X}$ having $\lambda(B) < \infty$. Then

$$D^n f(\mu) = \exp[-\mu(v)](e^{-v} - 1)^{\otimes n},$$

and the formula for the characteristic functional of Φ implies

$$T_n f = \exp[-\lambda(1 - e^{-v})](e^{-v} - 1)^{\otimes n}.$$

In particular $T_n f \in \mathbf{H}_n$, $n \geq 0$.

On the one hand we have

$$\mathbb{E}f(\eta)g(\eta) = \mathbb{E}\exp[-\eta(v+w)] = \exp[-\lambda(1-e^{-(v+w)})].$$

On the other hand,

$$\sum_{n=0}^{\infty}\frac{1}{n!}\langle T_n f, T_n g\rangle_n = \exp[-\lambda(1-e^{-v})]\exp[-\lambda(1-e^{-w})]\times$$
$$\times\sum_{n=0}^{\infty}\frac{1}{n!}\lambda^n(((e^{-v}-1)(e^{-w}-1))^{\otimes n})$$
$$= \exp[-\lambda(2-e^{-v}-e^{-w})]\exp[\lambda((e^{-v}-1)(e^{-w}-1))]$$
$$= \exp[-\lambda(1-e^{-(v+w)})].$$

The proof can now be accomplished as follows:

- The set **G** of all linear combinations of functions of the above type is dense in $L^2(\mathbb{P}_\eta)$.
- Use Hilbert space and completeness arguments to extend the result from **G** to $L^2(\mathbb{P}_\eta)$.

3. Absolute continuity of Poisson processes

Setting

For any σ-finite measure λ on $\mathbb{X}$ let $\mathbb{P}_\lambda$ be a probability measure governing a Poisson process Φ with intensity measure λ.

Theorem

Let ν, ρ be finite measures with $\nu \ll \rho$ and let h be the density. Then $\mathbb{P}_\nu(\Phi \in d\varphi) = L_{\nu,\rho}(\varphi)\mathbb{P}_\rho(\Phi \in d\varphi)$, where

$$L_{\nu,\rho}(\varphi) := \mathbf{1}\{\varphi(\mathbb{X}) < \infty\} e^{\rho(\mathbb{X})-\nu(\mathbb{X})} \prod_{y \in \operatorname{supp} \varphi} h(y)^{\varphi\{y\}}.$$

Theorem

Let ν, ρ be σ-finite measures with $\nu \ll \rho$ and let h be the density. Then $\mathbb{P}_\nu(\Phi \in \cdot) \ll \mathbb{P}_\rho(\Phi \in \cdot)$ if

$$\int (\sqrt{h} - 1)^2 d\rho < \infty.$$

Otherwise $\mathbb{P}_\nu(\Phi \in \cdot)$ and $\mathbb{P}_\rho(\Phi \in \cdot)$ are mutually singular.

Idea of the proof:

- If ν and λ are finite measures, then

$$\mathbb{E}\sqrt{L_{\nu,\rho}(\Phi)} = \exp\Big[-\frac{1}{2} \int (\sqrt{h} - 1)^2 d\rho \Big].$$

- Take a measurable partition C_n, $n \in \mathbb{N}$, of $\mathbb{X}$ with $\nu(C_n) < \infty$ and $\rho(C_n) < \infty$. Then

$$\mathbb{P}_\nu(\Phi_{C_n} \in d\varphi) = L_n(\varphi)\mathbb{P}_\rho(\Phi_{C_n} \in d\varphi),$$

where the function L_n satisfies

$$\mathbb{E}_\rho \sqrt{L_n(\Phi)} = \exp\Big[-\frac{1}{2} \int_{C_n} (\sqrt{h} - 1)^2 d\rho \Big].$$

- By Kakutani's (1948) dichotomy, $\mathbb{P}_\nu(\Phi \in \cdot) \ll \mathbb{P}_\rho(\Phi \in \cdot)$ if and only if

$$\prod_{n=1}^{\infty} \mathbb{E}_\rho \sqrt{L_n(\Phi)} > 0.$$

Remark

If $\int (h-1)^2 d\rho < \infty$ then $\int (\sqrt{h}-1)^2 < \infty$. In this case the density $L_{\nu,\rho}$ of $\mathbb{P}_\nu(\Phi \in \cdot)$ w.r.t. $\mathbb{P}_\rho(\Phi \in \cdot)$ satisfies

$$\mathbb{E}_\rho L_{\nu,\rho}(\Phi)^2 < \infty.$$

4. Perturbation analysis

Theorem (Finite perturbations, Molchanov and Zuyev '00)

Assume that μ is a finite measure and let $f : \mathbf{N} \to \mathbb{R}$ be such that $\mathbb{E}_{\lambda+\mu}|f(\Phi)| < \infty$. Then

$$\mathbb{E}_{\lambda+\mu} f(\Phi) = \mathbb{E}_{\lambda} f(\Phi) + \sum_{n=1}^{\infty} \frac{1}{n!} \int (\mathbb{E}_{\lambda} D^n_{x_1,\dots,x_n} f(\Phi)) \mu^n(d(x_1,\dots,x_n)),$$

where all expectations exist and the series converges absolutely.

Theorem (General perturbations)

Let λ and ν be two σ-finite measures on $\mathbb{X}$ and ρ a σ-finite measure dominating $\lambda + \nu$. Assume that

$$\int (1 - h_\lambda)^2 d\rho + \int (1 - h_\nu)^2 d\rho < \infty,$$

where $h_\lambda := d\lambda/d\rho$ and $h_\nu := d\nu/d\rho$. Let $f : \mathbf{N} \to \mathbb{R}$ be such that $\mathbb{E}_\rho f(\Phi)^2 < \infty$. Then

$$\mathbb{E}_\nu f(\Phi) = \mathbb{E}_\lambda f(\Phi) + \sum_{n=1}^{\infty} \frac{1}{n!} \int (\mathbb{E}_\lambda D^n_{x_1,\dots,x_n} f(\Phi))(\nu - \lambda)^n (d(x_1, \dots, x_n)),$$

where all integrals exist and the series converges absolutely.

Idea of the proof:

- The right-hand side of the asserted equation equals

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int (\mathbb{E}_\rho L_{\lambda,\rho}(\Phi) D^n_{x_1,\dots,x_n} f(\Phi))(\nu - \lambda)^n (d(x_1,\dots,x_n)),$$

  where $\mathbb{P}_\lambda(\Phi \in d\varphi) = L_{\lambda,\rho}(\varphi)\mathbb{P}_\rho(\Phi \in d\varphi)$.
- Treat $\mathbb{E}_\rho L_{\lambda,\rho}(\Phi) D^n_{x_1,\dots,x_n} f(\Phi)$ using the Fock space representation and the identity

$$\mathbb{E}_\rho D^k_{x_1,\dots,x_k} L_{\lambda,\rho}(\Phi) = \prod_{i=1}^{k} (h_\lambda(x_i) - 1).$$

Corollary

Let $\nu = \nu_1 + \nu_2$ be the Lebesgue decomposition of ν with respect to λ. Hence $\nu_1 \ll \lambda$ and $\nu_2 \perp \lambda$. Assume that

$$\int (1 - d\nu_1/d\lambda)^2 d\lambda + \nu_2(\mathbb{X}) < \infty.$$

Let $f : \mathbf{N} \to \mathbb{R}$ be such that $\mathbb{E}_{\lambda+\nu_2} f(\Phi)^2 < \infty$ Then the variational formula holds.

Corollary

Let $\lambda = \lambda_1 + \lambda_2$ be the Lebesgue decomposition of λ with respect to ν and assume that

$$\int (1 - d\lambda_1/d\nu)^2 d\nu + \lambda_2(\mathbb{X}) < \infty.$$

Let $f : \mathbf{N} \to \mathbb{R}$ be such that $\mathbb{E}_{\nu+\lambda_2} f(\Phi)^2 < \infty$. Then the variational formula holds.

Theorem

Let C_1, C_2 be disjoint measurable sets such that λ and ν are equivalent on C_1 and orthgonal on C_2. If the variational formula holds for all bounded functions f then

$$\int_{C_1} (1 - \sqrt{d\nu/d\lambda})^2 d\lambda + \lambda(C_2) + \nu(C_2) < \infty. \tag{1}$$

In this case the distributions $\mathbb{P}_\lambda(\Phi_{C_1} \in \cdot)$ and $\mathbb{P}_\nu(\Phi_{C_1} \in \cdot)$ are equivalent.

Remark

If the density in (1) or its inverse is bounded on C_1 then (1) is also sufficient for the variational formula to hold for all bounded measurable functions f.

5. Derivatives

Theorem

Let λ and ν be as above and let $f : \mathbf{N} \to \mathbb{R}$ be a measurable function such that $\mathbb{E}_\rho f(\Phi)^2 < \infty$. Then

$$\mathbb{E}_{\lambda+t(\nu-\lambda)} f(\Phi) = \mathbb{E}_\lambda f(\Phi) + \sum_{n=1}^{\infty} \frac{t^n}{n!} \int (\mathbb{E}_\lambda D^n_{x_1,\dots,x_n} f(\Phi))(\nu-\lambda)^n(d(x_1,\dots,x_n)),$$

where the series converges absolutely for all $t \in [0,1]$. Moreover,

$$\frac{d}{dt}\mathbb{E}_{\lambda+t(\nu-\lambda)} f(\Phi) = \int \mathbb{E}_{\lambda+t(\nu-\lambda)} D_x f(\Phi)(\nu-\lambda)(dx).$$

Theorem (Margulis-Russo type formula, Decreusefond '98)

Assume that $\int h^2 d\lambda < \infty$. *Let* $I \subset \mathbb{R}$ *be an interval with non-empty interior and* $\theta_0 \in I$. *For any* $\theta \in I$ *let* $R_\theta : \mathbb{X} \to \mathbb{R}$ *be a measurable function such that the following assumptions are satisfied:*

- *For all* $\theta \in I$, $1 + (\theta - \theta_0)(h + R_\theta) \geq 0$ λ*-a.e.*
- $\lim_{\theta \to \theta_0} \int R_\theta^2 d\lambda = 0$.

For $\theta \in I$, *let* λ_θ *denote the measure with density* $1 + (\theta - \theta_0)(h + R_\theta)$ *with respect to* λ. *Let* $f : \mathbf{N} \to \mathbb{R}$ *be a measurable function such that* $\mathbb{E}_\lambda f(\Phi)^2 < \infty$. *Then*

$$\frac{d}{d\theta} \mathbb{E}_{\lambda_\theta} f(\Phi) \Big|_{\theta = \theta_0} = \int (\mathbb{E}_\lambda D_x f(\Phi)) h(x) \lambda(dx).$$

6. Perturbation analysis of Lévy processes

Definition

A Lévy process $(X_t)_{t \geq 0}$ is a $\mathbb{R}^d$-valued process with $X_0 = 0$ and independent and stationary increments that is right-continuous and has left-hand limits.

Definition

A *Lévy measure* is a measure on $\mathbb{R}^d$ with $\nu(\{0\}) = 0$, and

$$\int (|x|^2 \wedge 1)\nu(dx) < \infty.$$

Theorem (Lévy-Khinchin representation)

A Lévy process $(X_t)_{t\geq 0}$ can be represented as

$$X_t = bt + \sigma B_t + \int_{|x|\leq 1}\int_0^t x(\Phi(ds,dx) - ds\nu(dx)) + \int_{|x|>1}\int_0^t x\Phi(ds,dx),$$

where $b \in \mathbb{R}^d$, σ is a $d \times d$-matrix, $(B_t)_{t\geq 0}$ is a Brownian motion in $\mathbb{R}^d$, and Φ is a Poisson process on $[0,\infty)\times\mathbb{R}^d$ with intensity measure $\lambda_1 \otimes \nu$, independent of (B_t).

Setting

We fix a rcll process $X = (X_t)_{t\geq 0}$ and $\sigma \geq 0$, and let $\mathbb{P}_{b,\nu}$ denote a probability measure such that $\mathbb{P}_{b,\nu}(X \in \cdot)$ is the distribution of a Lévy process with characteristic triplet $(\sigma\sigma', b, \nu)$.

Definition

Let **D** denote the space of all $\mathbb{R}^d$-valued rcll functions on $[0, \infty)$ equipped with the product σ-field.

Definition (Difference operator)

For $w \in \mathbf{D}$ and $(t_1, x_1) \in [0, \infty) \times \mathbb{R}^d$ define $w^{t_1,x_1} \in \mathbf{D}$ by $w^{t_1,x_1}_t := w_t + \mathbf{1}\{t \geq t_1\}x_1$. For any measurable $f : \mathbf{D} \to \mathbb{R}$ the measurable function $\Delta_{t_1,x_1} f : \mathbf{D} \to \mathbb{R}$ is defined by

$$\Delta_{t_1,x_1} f(w) := f(w^{t_1,x_1}) - f(w), \quad w \in \mathbf{D}.$$

Similarly as before one can iterate this definition to obtain, for $(t_1, x_1, \dots, t_n, x_n) \in ([0, \infty) \times \mathbb{R}^d)^n$ a function $\Delta^n_{t_1,x_1,\dots,t_n,x_n} f : \mathbf{D} \to \mathbb{R}$.

Setting

We consider three Lévy measures ν, ν', ν^* such that ν and ν' are absolutely continuous with respect to ν^* with densities g_ν and $g_{\nu'}$, respectively, that satisfy

$$\int (1 - g_\nu)^2 d\nu^* + \int (1 - g_{\nu'})^2 d\nu^* < \infty. \tag{2}$$

We also consider $b, b', b^* \in \mathbb{R}^d$ such that

$$b = b^* + \int \mathbf{1}\{|x| \le 1\} x(\nu - \nu^*)(dx), \tag{3}$$

$$b' = b^* + \int \mathbf{1}\{|x| \le 1\} x(\nu' - \nu^*)(dx). \tag{4}$$

Theorem

Assume that (2)-(4) *hold. Let* $f : \mathbf{D} \to \mathbb{R}$ *be measurable and assume that* $f(w)$ *depends only on the restriction of* $w \in \mathbf{D}$ *to some finite interval. If* $\mathbb{E}_{b^*,\nu^*} f(X)^2 < \infty$*, then*

$$\mathbb{E}_{b',\nu'} f(X) = \mathbb{E}_{b,\nu} f(X) + \sum_{n=1}^{\infty} \frac{1}{n!} \int (\mathbb{E}_{b,\nu} \Delta^n f(X))(g_{\nu'} - g_\nu)^{\otimes n} d(\lambda_1 \otimes \nu^*)^n,$$

where $\mathbb{E}_{b,\nu} \Delta^n f(X)$ *denotes the function* $(t_1, x_1, \ldots, t_n, x_n) \mapsto \mathbb{E}_{b,\nu} \Delta^n_{t_1,x_1,\ldots,t_n,x_n} f(X)$*.*

Setting

Consider a Lévy measure ν and a measurable function $g : \mathbb{R}^d \to \mathbb{R}$ such that

$$\int g(x)^2 \nu(dx) < \infty.$$

Let $I \subset \mathbb{R}$ be an interval with non-empty interior and $\theta_0 \in I$. For $\theta \in I$ let $R_\theta : \mathbb{R}^d \to [0, \infty)$ be a measurable function such that:

(i) For all $\theta \in I$, $1 + (\theta - \theta_0)(g + R_\theta) \geq 0$ ν-a.e.

(ii) $\int (|x| \wedge 1)|R_\theta(x)|\nu(dx) < \infty$.

(iii) $\lim_{\theta \to \theta_0} \int R_\theta^2 d\nu = 0$.

Theorem

For $\theta \in I$, let ν_θ denote the measure with density $1 + (\theta - \theta_0)(g + R_\theta)$ with respect to ν. Let $b \in \mathbb{R}$ and define

$$b_\theta := b + (\theta - \theta_0) \int \mathbf{1}\{|x| \leq 1\} x(g(x) + R_\theta(x))\nu(dx), \quad \theta \in I.$$

Let $f : \mathbf{D} \to \mathbb{R}$ be a measurable function with finite support such that $\mathbb{E}_{b,\nu} f(X)^2 < \infty$. Then

$$\frac{d}{d\theta} \mathbb{E}_{b_\theta, \nu_\theta} f(X) \Big|_{\theta=\theta_0} = \iint (\mathbb{E}_{b,\nu} \Delta_{t,x} f(X)) g(x) dt \nu(dx).$$

7. References

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