Yaglom limit via Holley inequality

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The p-q discrete time random walk on $\{0\} \cup \mathbb{N}$ absorbed at 0.

$$Q(x, x+1) = p, \quad Q(x, x-1) = q, \quad Q(0,0) = 1$$

$p < q$.

Conditioned distribution:

Initial distribution ν, a probability on $\mathbb{N}$.

Distribution of walk conditioned to stay in $\mathbb{N}$ during $[0, n]$:

$$\nu T_n(y) := \frac{\nu Q^n(y)}{1 - \nu Q^n(0)}, \qquad y \in S. \tag{1}$$

Def: ν is a *quasi stationary distribution* (qsd) if

$$\nu T_n = \nu, \qquad n \geq 1.$$

Absorption time of qsd is exponential:

$$P(\tau^\nu > t) = e^{-R(\nu)}$$

There are infinitely many qsd ordered by absorption rate

$$R(\nu) = q\nu(1) \in [0, q(1-\sqrt{\lambda})^2], \qquad \lambda = p/q.$$

The minimal qsd $\nu_{\min}$ is negative binomial $(2, \sqrt{\lambda})$:

$$\nu_{\min}(x) = \big(1-\sqrt{\lambda}\big)^2 x \big(\sqrt{\lambda}\big)^{x-1}, \quad x \geq 1. \tag{2}$$

and the others are given in function of $\nu(1) < \big(1-\sqrt{\lambda}\big)^2$ by

$$\nu(x) = \frac{\nu(1)}{c}\Big[\Big(\frac{\lambda+1-\nu(1)+c}{2}\Big)^x - \Big(\frac{\lambda+1-\nu(1)-c}{2}\Big)^x\Big] \tag{3}$$

where $c = [(\nu(1)-\lambda-1)^2 - 4\lambda]^{1/2}$. See Cavender [1], pag 585.

The *Yaglom limit* of ν is

$$\lim_n \nu T_n,$$

if the limit exists and is a probability.

Stochastic domination:

$\mathbb{N}$ is well ordered with minimal state 1: $1 \leq x$ for all $x \in \mathbb{N}$.

$\nu \prec \nu'$ if and only if $\nu f \leq \nu' f$ for all non decreasing $f : \mathbb{N} \to \mathbb{R}$

Coupling: $\nu \prec \nu'$ if and only if there exists coupling $\tilde{\nu}$ on $\mathbb{N} \times \mathbb{N}$ with marginals ν and ν' such that $\tilde{\nu}((x, x') : x \leq x') = 1$.

Let δ_1 measure concentrating mass on 1.

Interested in Yaglom limit starting from δ_1:

$$\lim_n \delta_1 T_{2n}, \qquad \lim_n \delta_1 T_{2n+1}$$

Period 2: starting from 1, visits odd sites at even times and even sites at odd times.

$\nu(\cdot|\mathrm{odd})$ be the measure ν conditioned to odd values

$\nu(\cdot|\mathrm{even})$, conditioned to even values.

If ν is qsd, then

$$\nu(\cdot|\mathrm{odd})\, T_{2n} = \nu(\cdot|\mathrm{odd}), \qquad \nu(\cdot|\mathrm{odd})\, T_{2n+1} = \nu(\cdot|\mathrm{even})$$

Theorem 1

i. *The sequence of measures* $(\delta_1 T_{2n}, n \geq 0)$ *is monotone:*

$$\delta_1 T_{2n} \prec \delta_1 T_{2n+2}, \qquad \text{for all } n \geq 0.$$

ii. *If* ν *is a qsd, then*

$$\delta_1 T_{2n} \prec \nu(\cdot|\mathrm{odd}), \qquad \delta_1 T_{2n+1} \prec \nu(\cdot|\mathrm{even})$$

iii. *Yaglom limit selects minimal qsd:*

$$\lim_n \delta_1 T_{2n} = \nu_{\min}(\cdot|\mathrm{odd}), \qquad \lim_n \delta_1 T_{2n+1} = \nu_{\min}(\cdot|\mathrm{even})$$

and $\nu_{\min} \prec \nu$, *for any qsd* ν.

Background

Yaglom limit (iii) proven by Seneta, Seneta and Vere Jones, Van Doorn and Schrijner using explicit calculations.

Trajectory distribution

For time integers $n < m$, trajectories in $\mathbb{N}$:

$$\mathbb{N}_n^m := \{x_n^m = (x_n, \ldots, x_m) : x_k \in S,\ k = n, \ldots, m\}$$

Define

$$\mu_n^m(\nu, Q)(x_n^m) := \frac{\nu(x_n) Q(x_n, x_{n+1}) \ldots Q(x_{m-1}, x_m)}{1 - \nu Q^{m-n}(0)} \tag{4}$$

Distribution of chain $X_n^m = (X_n, \ldots, X_m)$ with initial distribution $P(X_n = \cdot) = \nu$ conditioned to stay in S during $[n, m]$.

$$\nu T_{m-n}(y) = \sum_{(x_n, \ldots, x_{m-1}) \in \mathcal{X}_n^{m-1}} \mu_n^m(\nu, Q)(x_n, \ldots, x_{m-1}, y). \tag{5}$$

The m-th marginal of $\mu_n^m(\nu, Q)$ has distribution νT_{m-n}.

Domination

Partial order on $\mathbb{N}_n^m$ is coordinatewise order of trajectories:

$$x_n^m \leq y_n^m \text{ if } x_k \leq y_k \text{ for all } k \in [n, m].$$

Order of measures on $\mathbb{N}_n^m$:

$\mu \prec \mu'$ iff there is a coupling $\tilde{\mu}$ with marginals μ, μ' such that

$\tilde{\mu}(x_n^m \leq y_n^m)$

Since we start with δ_1, we work in the space

$$(\mathbb{N}_n^m)_{\mathrm{odd}} = \{x_n^m \in \mathbb{N}_n^m : x_k \in 2\mathbb{N} + \mathbf{1}\{k - n \text{ is even}\},\ k \in [n, m]\}$$

and $\delta_1 T_{m-n}(x) = 0$ if $m - n + x$ is even.

(Simple version of) Holley inequality:

Proposition Let ν be a probability on $(\mathbb{N}_n^m)_{\mathrm{odd}}$. Then,

$$\mu_n^m(\delta_1, Q) \prec \mu_n^m(\nu, Q).$$

Gibbs sampler on the state space of trajectories:

Continuous time Markov chain with rates: $n < k < m$:

$k-1$	k	$k+1$		$k-1$	k	$k+1$	
x	$*$	x	$\rightarrow$	x	$x+1$	x	rate 1
x	$*$	x	$\rightarrow$	x	$x-1$	x	rate 1 if $x > 1$

$m-1$	m		$m-1$	m	
x	$*$	$\rightarrow$	x	$x+1$	rate p
x	$*$	$\rightarrow$	x	$x-1$	rate q if $x > 1$

n	$n+1$		n	$n+1$	
$*$	x	$\rightarrow$	$x+1$	x	rate $\frac{q\nu(x+1)}{q\nu(x+1)+p\nu(x-1)}$
$*$	x	$\rightarrow$	$x-1$	x	rate $\frac{p\nu(x-1)}{q\nu(x+1)+p\nu(x-1)}$

"ν boundary conditions"

$\mu(\nu, Q)$ is reversible for Gibbs sampler with ν boundary conditions.

Substituting the left boundary condition by $x_n \equiv 1$:

$\mu(\delta_1, Q)$ is reversible for Gibbs sampler with δ_1 boundary conditions.

Coupling $((\eta_\ell, \eta'_\ell) : \ell \in \mathbb{N})$ on $\mathcal{X}_n^m \times \mathcal{X}_n^m$

Use the same Poisson clocks to update both marginals with the rates above:

First marginal with boundary condition δ_1.

Second marginal with boundary condition ν.

Hence marginals are Gibbs sampler for $\mu = \mu(\delta_1, Q)$ and $\mu' = \mu(\nu, Q)$, respectively.

The coupling is monotone: $\eta_0 \leq \eta'_0$ implies $\eta_\ell \leq \eta'_\ell$ for all $\ell \geq 0$.

Proof of Holley inequality

Define $\underline{1} = (121\ldots121)$ minimal configuration in $(\mathbb{N}_n^m)_{\rm odd}$

Start $(\eta_0, \eta'_0) = (\underline{1}, \underline{1})$.   Call $\tilde{\mu}_\ell$: law of (η_ℓ, η'_ℓ).

$\tilde{\mu}_\ell(\eta \le \eta') = 1$ for all $\ell \ge 0$ (monotonicity).

Process is attractive $\tilde{\mu}_\ell$ is stochastically non decreasing.

Each marginal converges to the respective invariant measure:

$$\mu_\ell \nearrow \mu, \qquad \mu'_\ell \nearrow \mu'$$

$\tilde{\mu}_\ell \nearrow \tilde{\mu}$, an invariant measure for the coupled process.

$\tilde{\mu}$ concentrates on $\eta \le \eta'$. Hence $\mu \prec \mu'$. $\square$

Monotonicity and Yaglom limit

Proof of Theorem 1

Proof of i. Modification of proof of Holley gives

$$\mu^0_{-n}(\delta_1, Q) \prec \mu^0_{-n-1}(\delta_1, Q). \tag{6}$$

Hence, the corresponding 0-marginals are also ordered:

$$\delta_1 T_n \prec \delta_1 T_{n+1}.$$

Proof of ii. Let ν' be a qsd. By Holley:

$$\mu^0_{-n}(\delta_1, Q) \prec \mu^0_{-n}(\nu', Q)$$

which implies $\delta_1 T_n \prec \nu' T_n = \nu'$, because ν' is qsd.

Proof of iii. Denote $\nu_n = \delta_1 T_n$ and let ν' be a qsd.

By (i) ν_n is an increasing sequence of measures.

By (*ii*), $\nu_n \prec \nu'$, for all $n \geq 0$.

Hence there is a limit $\nu = \lim_n \nu_n \prec \nu'$.

To check that ν is a qsd, follows from (1) that

$$\nu_{n+1}(y) = \sum_x \nu_n(x)\big(Q(x,y) + Q(x,0)\nu_{n+1}(y)\big)$$

Hence $\lim_n \nu_n$ must satisfy equation

$$\nu(y) = \sum_x \nu(x)\big(Q(x,y) + Q(x,0)\nu(y)\big)$$

characterizing a qsd. $\square$

General Setup

S partial ordered set with minimal element 1.

Theorem 1 in general *Assume that Q is the transition matrix of a irreducible aperiodic Markov chain on $S \cup \{0\}$ absorbed at 0 such that there is at least one qsd for Q and*

$$\frac{Q(x,\cdot)Q(\cdot,z)}{Q^2(x,z)} \prec \frac{Q(x',\cdot)Q(\cdot,z')}{Q^2(x',z')}, \tag{7}$$

$$\frac{Q(x,\cdot)}{1-Q(x,0)} \prec \frac{Q(x',\cdot)}{1-Q(x',0)}, \tag{8}$$

for all $z, z', x, x' \in S$ such that $z \le z'$, $x \le x'$. Then,

i. *The sequence of measures* $(\delta_1 T_n, n \geq 1)$ *is monotone:*

$$\delta_1 T_n \prec \delta_1 T_{n+1}, \qquad \textit{for all } n \geq 0. \tag{9}$$

ii. *If* ν *is a qsd, then*

$$\delta_1 T_n \prec \nu, \qquad \textit{for all } n \geq 0. \tag{10}$$

iii. *The Yaglom limit of* δ_1 *converges to a qsd denoted* $\nu_{\min}$*:*

$$\lim_n \delta_1 T_n = \nu_{\min} \tag{11}$$

and $\nu_{\min} \prec \nu$*, for any qsd* ν*.*

Proposition (Holley inequality) *Let ν, ν' be probabilities on S and Q, Q' be transition matrices on $S \cup \{0\}$ absorbed at 0 such that*

(a) $\dfrac{\nu(\cdot)Q(\cdot,z)}{\nu Q(z)} \prec \dfrac{\nu'(\cdot)Q'(\cdot,z')}{\nu' Q'(z')}$, *for all $z \leq z'$, with $z, z' \in S$;*

(b) $\dfrac{Q(x,\cdot)Q(\cdot,z)}{Q^2(x,z)} \prec \dfrac{Q'(x',\cdot)Q(\cdot,z')}{Q'^2(x',z')}$, *for all $x \leq x'$, $z \leq z'$, with $x, x', z, z' \in S$;*

(c) $Q(x,\cdot) \prec Q'(x',\cdot)$, *for all $x \leq x'$, with $x, x' \in S$.*

Assume also that both $\mu_n^m(\nu, Q)$ and $\mu_n^m(\nu', Q')$ are irreducible probability measures on $\mathcal{X}_n^m$. Then,

$$\mu_n^m(\nu, Q) \prec \mu_n^m(\nu', Q').$$

One-dimensional examples

Random walk with delay The absorbed delayed random walk:

Parameters $p, q, r > 0$; $p < q$; $p + q + r = 1$.

Transition probabilities:

$$\begin{aligned} &Q(x, x-1) = q, \quad Q(x, x) = r \ \text{(delay)}, \quad Q(x, x+1) = p, \\ &Q(0,0) = 1, \quad Q(x, y) = 0, \ \text{otherwise}, \quad x \geq 1. \end{aligned} \tag{12}$$

Drift towards 0 and absorbed at 0.

Irreducible aperiodic random walk on $\mathbb{N} \cup \{0\}$.

The qsd are the same as for the p-q random walk.

Holley conditions (b,c) are satisfied if $pq \leq r^2$ and we get:

Theorem *For the delayed random walk with $pq \leq r^2$ the conclusions (i, ii, iii) of Theorem 1 hold.*

The continuous time random walk

Positive real numbers $p < q$

Family of random walks with delay (X_n^r), indexed by r (large):

$$Q_r(x, x-1) = q(1-r),\ Q_r(x,x) = r,\ Q_r(x, x+1) = p(1-r),$$

$$Q_r(x,y) = 0,\ \text{otherwise};$$

for $x \geq 1$; $Q_r(0,0) = 1$. r. Rescaled process:

$$Y_t^r := X_{[t/(1-r)]}^r$$

As r goes to 1, (Y_t^r) converges to $(\hat{Y}_t)$, a continuous time random walk with rates p, q absorbed at 0, with semigroup $\hat{U}_t$ given by:

$$\hat{U}_t(x,y) := P(\hat{Y}_t = y | \hat{Y}_0 = x).$$

Define the conditioned delayed evolution as before:

$$\nu T_t^r := \frac{\nu Q_r^{[t/(1-r)]}(y)}{1 - \nu Q_r^{[t/(1-r)]}(0)}$$

And, in the limit, the continuous conditioned evolution by:

$$\lim_{r\to 1} \nu T_t^r := \nu \hat{T}_t(y) = \frac{\nu \hat{U}_t(y)}{1 - \nu \hat{U}_t(0)} \tag{13}$$

Theorem *The continuous time random walk with rates p, q absorbed at zero satisfies*

i. *The sequence $(\delta_1 \hat{T}_t, n \geq 1)$ is monotone: $\delta_1 \hat{T}_t \prec \delta_1 \hat{T}_{t+s}$ for $t, s \geq 0$.*
ii. *If ν is a qsd, then $\delta_1 \hat{T}_t \prec \nu$ for all $t \geq 0$.*

iii. *The Yaglom limit of* δ_1 *converges to* $\nu_{\min}$ *given by* (2)*:* $\lim_n \delta_1 \hat{T}_t = \nu_{\min}$.

Item (iii) was proven by direct computation by Seneta [7]. Our proof is a consequence of monotonicity:

Proof (i) Use part (i) of the delayed Theorem with $r > 1/2$ to get

$$\delta_1 T_t^r \prec \delta_1 \hat{T}_{t+s}^r, \qquad \text{for } t, s \geq 0$$

and use (13) to conclude.

(ii) Use the fact that the qsd for Y_t^r are the same as the qsd for $\hat{Y}_t$ and Theorem 1(ii) to conclude $\delta_1 \hat{T}_t = \lim_{r\to 1} \delta_1 T_t^r \prec \nu$.

(iii) is consequence of (i,ii) like in the proof of Theorem 1. $\square$

Question Can one prove Holley inequality in the continuous time case without the discrete limit?.

One should devise a reversible attractive dynamics for which the law of the continuous-time trajectories in finite intervals conditioned to stay positive is reversible.

Brownian motion

X_t^ε is random walk with no delay and probabilities

$p = \frac{a}{2} - \varepsilon$, $q = \frac{a}{2} + \varepsilon$

$Z_t^\varepsilon := \varepsilon X_{\varepsilon^{-2}t}^\varepsilon$ (diffusively rescaled random walk with drift)

(Z_t^ε) converges to Brownian motion with drift $(B_t + at)$

Holley inequality holds for $(X_t^\varepsilon, t \in [0, \bar{t}])$ (fixed ε and $\bar{t}$).

Two possibilities:

1) show the inequalities for fixed ε and show that the conditioned trajectories of Z^ε converge to the conditioned trajectories of B_t

2) Define a "limiting dynamics" directly on the trajectories of BM to show Holley inequality for the conditioned trajectories.

(Under construction)

Final remarks

- There are some two-dimensional examples.
- Is it possible to relax the condition of only one minimal state?
- Attractiveness far for absorption implies condition (7)?
- If process without conditioning is attractive, then $\nu_{\min}$ has minimal expected absorption time.
- Attractive dynamics guarantee δ_1 has minimal expected absorption time?

Open problem

(X_t, Y_t) queues in series.

$X_t =$ number of customers in queue 1 at time t

$Y_t =$number of customers in queue 2 at time t

Customers enter queue 1 at rate $\rho < 1$

Service is exponential at rate 1 in both queues.

Customers served at queue 1 jump to queue 2.

The process is absorbed when queue 2 is empty: $Y_y = 0$.

(minimal?) qsd: $\nu(x, y) = C\rho^x y \rho^{y/2}$, $x \geq 0$, $y > 0$

product of geometric and negative binomial.

Problem: Prove that the Yaglom limit starting from $\delta_{(0,1)}$ converges to ν.

References

Cavender [1]

Van Doorn and Schrijner [9] [10]

Ferrari, Martínez and Picco [5] [4]

Collet, Martinez y San Martin [2]

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