Variance asympotics and scaling limits for Gaussian polytopes

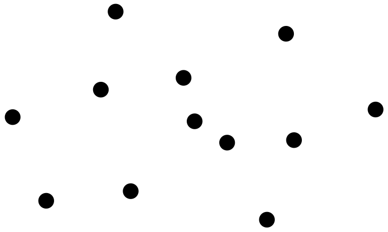
Joe Yukich (joint with Pierre Calka)

Lehigh University

Simons Workshop on Stochastic Geometry and Point Processes

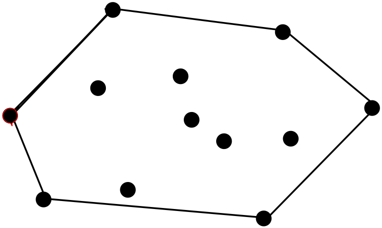
Introduction

random points:



Introduction

convex hull of random points:



extreme points = vertices

Historical remarks

random points $X_1, ..., X_4$ in $K \subset \mathbb{R}^2$; $\quad K_4 = \text{conv}[X_1, ..., X_4] \subset \mathbb{R}^2$

$$f_0(K_4) = \text{number of vertices of } K_4 = ?$$

April 1864, Educational Times, J. J. Sylvester (1814 - 1897)

Historical remarks

$X_1, ..., X_4$ random points in $K \subset \mathbb{R}^2$.

$K = \Delta \quad \mathbb{E}\, f_0(K_4) = 11/3$ (Sylvester)

Historical remarks

$X_1, ..., X_4$ random points in $K \subset \mathbb{R}^2$.

$K = \Delta \quad \mathbb{E}\, f_0(K_4) = 11/3$ (Sylvester)

$K = B^2 \quad \mathbb{E}\, f_0(K_4) = \frac{48\pi^2 - 35}{12\pi^2}$ (Woolhouse)

Historical remarks

$X_1, ..., X_4$ random points in $K \subset \mathbb{R}^2$.

$K = \Delta \quad \mathbb{E}\, f_0(K_4) = 11/3$ (Sylvester)

$K = B^2 \quad \mathbb{E}\, f_0(K_4) = \frac{48\pi^2 - 35}{12\pi^2}$ (Woolhouse)

$K = \square \quad \mathbb{E}\, f_0(K_4) = \frac{133}{36}$ (Woolhouse)

Historical remarks

$X_1, ..., X_4$ random points in $K \subset \mathbb{R}^2$.

$K = \Delta \quad \mathbb{E}\, f_0(K_4) = 11/3$ (Sylvester)

$K = B^2 \quad \mathbb{E}\, f_0(K_4) = \frac{48\pi^2 - 35}{12\pi^2}$ (Woolhouse)

$K = \square \quad \mathbb{E}\, f_0(K_4) = \frac{133}{36}$ (Woolhouse)

Alikoski, Blaschke, Crofton, Dalla, Efron, Groemer, Herglotz, Larman, Schneider

Historical remarks

$d \geq 3$:

$K = B^d \quad \mathbb{E}\, f_0(K_n) = \;\; ....$ (Buchta, Affentranger)

Historical remarks

$d \geq 3$:

$K = B^d \quad \mathbb{E}\, f_0(K_n) = \ \ldots.$ (Buchta, Affentranger)

$K = \square^3 \quad \mathbb{E}\, f_0(K_5) = \frac{212023}{43200} - \frac{\pi^2}{432}$ (Zinani)

$K = \Delta^3 \quad \mathbb{E}\, f_0(K_n) = \ \ldots$ (Buchta, Reitzner)

Inequalites

Blaschke (1917): for all compact convex $K \subset \mathbb{R}^2$

$$\mathbb{E}\, f_0(K_4^{\Delta}) \leq \mathbb{E}\, f_0(K_4^{K}) \leq \mathbb{E}\, f_0(K_4^{B^2}).$$

The triangle and disc are extremal when input is a four pt. set.

Inequalites

Blaschke (1917): for all compact convex $K \subset \mathbb{R}^2$

$$\mathbb{E}\, f_0(K_4^{\Delta}) \leq \mathbb{E}\, f_0(K_4^{K}) \leq \mathbb{E}\, f_0(K_4^{B^2}).$$

The triangle and disc are extremal when input is a four pt. set.

Dalla and Larman (1991) showed this holds for all $n \geq 4$, ie.

$$\mathbb{E}\, f_0(K_n^{\Delta}) \leq \mathbb{E}\, f_0(K_n^{K}) \leq \mathbb{E}\, f_0(K_n^{B^2}).$$

Inequalites

Blaschke (1917): for all compact convex $K \subset \mathbb{R}^2$

$$\mathbb{E}\, f_0(K_4^{\Delta}) \le \mathbb{E}\, f_0(K_4^{K}) \le \mathbb{E}\, f_0(K_4^{B^2}).$$

The triangle and disc are extremal when input is a four pt. set.

Dalla and Larman (1991) showed this holds for all $n \ge 4$, ie.

$$\mathbb{E}\, f_0(K_n^{\Delta}) \le \mathbb{E}\, f_0(K_n^{K}) \le \mathbb{E}\, f_0(K_n^{B^2}).$$

Proving the extremal property of the simplex in higher dimensions seems to be difficult and is open.

Surprising questions

(i) Is the expected vertex count monotone in input size? Do we have

$$\mathbb{E}\, f_0(K_n^K) \leq \mathbb{E}\, f_0(K_{n+1}^K)?$$

Surprising questions

(i) Is the expected vertex count monotone in input size? Do we have

$$\mathbb{E}\, f_0(K_n^K) \leq \mathbb{E}\, f_0(K_{n+1}^K)?$$

(ii) If $K, L \subset \mathbb{R}^d$ are convex, $K \subset L$, do we have

$$\mathbb{E}\,\mathrm{Vol}_d(K_n^K) \leq \mathbb{E}\,\mathrm{Vol}_d(L_n^L)?$$

Surprising questions

(i) Is the expected vertex count monotone in input size? Do we have

$$\mathbb{E}\, f_0(K_n^K) \leq \mathbb{E}\, f_0(K_{n+1}^K)?$$

(ii) If $K, L \subset \mathbb{R}^d$ are convex, $K \subset L$, do we have

$$\mathbb{E}\,\mathrm{Vol}_d(K_n^K) \leq \mathbb{E}\,\mathrm{Vol}_d(L_n^L)?$$

Rademacher $d \geq 4, n = d+1$: 'no'

Surprising questions

(i) Is the expected vertex count monotone in input size? Do we have

$$\mathbb{E}\, f_0(K_n^K) \le \mathbb{E}\, f_0(K_{n+1}^K)?$$

(ii) If $K, L \subset \mathbb{R}^d$ are convex, $K \subset L$, do we have

$$\mathbb{E}\,\mathrm{Vol}_d(K_n^K) \le \mathbb{E}\,\mathrm{Vol}_d(L_n^L)?$$

Rademacher $d \ge 4, n = d + 1$: 'no'

If $d = 2, 3$, then 'yes'.

Statistics of random polytopes

$K_n = \mathrm{conv}[X_1, ..., X_n] \subset K$

$f_0(K_n) =$ number of vertices of K_n

Statistics of random polytopes

$K_n = \mathrm{conv}[X_1, ..., X_n] \subset K$

$f_0(K_n) =$ number of vertices of K_n

$f_1(K_n) =$ number of edges of K_n

Statistics of random polytopes

$K_n = \text{conv}[X_1, ..., X_n] \subset K$

$f_0(K_n) =$ number of vertices of K_n

$f_1(K_n) =$ number of edges of K_n

$f_\ell(K_n) =$ number of ℓ-faces of K_n, $\ell \in \{0, ..., d-1\}$.

Statistics of random polytopes

$K_n = \mathrm{conv}[X_1, ..., X_n] \subset K$

$f_0(K_n) =$ number of vertices of K_n

$f_1(K_n) =$ number of edges of K_n

$f_\ell(K_n) =$ number of ℓ-faces of K_n, $\ell \in \{0, ..., d-1\}$.

$\mathrm{Vol}_d(K_n) =$ volume of K_n.

Expectation asymptotics

Difficult to derive explicit formula for statistics of convex hulls on finite point sets.

Investigation has focussed on behavior as input size $n \to \infty$.

Expectation asymptotics

Difficult to derive explicit formula for statistics of convex hulls on finite point sets.

Investigation has focussed on behavior as input size $n \to \infty$.

Expectation asymptotics

Difficult to derive explicit formula for statistics of convex hulls on finite point sets.

Investigation has focussed on behavior as input size $n \to \infty$.

K compact convex: $\mathbb{E}\, f_\ell(K_n)$ tends to infinity, $\ell \in \{0, ..., d-1\}$.

The shape of ∂K determines the order of magnitude of $\mathbb{E}\, f_\ell(K_n)$.

Expectation asymptotics

Difficult to derive explicit formula for statistics of convex hulls on finite point sets.

Investigation has focussed on behavior as input size $n \to \infty$.

K compact convex: $\mathbb{E}\, f_\ell(K_n)$ tends to infinity, $\ell \in \{0, ..., d-1\}$.

The shape of ∂K determines the order of magnitude of $\mathbb{E}\, f_\ell(K_n)$.

Affine surface area: $\int_{\partial K} \kappa(x)^{1/(d+1)} dx$.

$\kappa(x)$: Gaussian curvature at $x \in \partial K$ (product of principal curvatures).

Expectation asymptotics (d = 2)

Rényi and Sulanke (1963-64), X_i i.i.d. in K, ∂K smooth ($d = 2$):

$$\lim_{n \to \infty} n^{-1/3} \mathbb{E}\, f_0(K_n) = e_{0,d} (\mathrm{Vol} K)^{-1/3} \int_{\partial K} \kappa(x)^{1/3} dx$$

Expectation asymptotics ($d \geq 2, \mathrm{Vol} K = 1$)

- Reitzner (2005): ∂K of class C^2, $\ell \in \{0, 1, ..., d-1\}$, $d \geq 2$:

$$\lim_{n \to \infty} n^{-(d-1)/(d+1)} \mathbb{E}\, f_\ell(K_n) = e_{\ell,d} \int_{\partial K} \kappa(x)^{1/(d+1)} dx.$$

Expectation asymptotics ($d \geq 2, \mathrm{Vol} K = 1$)

- Reitzner (2005): ∂K of class C^2, $\ell \in \{0, 1, ..., d-1\}$, $d \geq 2$:

$$\lim_{n \to \infty} n^{-(d-1)/(d+1)} \mathbb{E}\, f_\ell(K_n) = e_{\ell,d} \int_{\partial K} \kappa(x)^{1/(d+1)} dx.$$

- K is a convex polytope, $\ell \in \{0, 1, ..., d-1\}$, $d \geq 2$:

$$\lim_{n \to \infty} (\log n)^{-(d-1)} \mathbb{E}\, f_\ell(K_n) = e'_{\ell,d} \cdot \text{number of flags of K}.$$

Expectation asymptotics ($d \geq 2, \text{Vol}K = 1$)

- Reitzner (2005): ∂K of class C^2, $\ell \in \{0, 1, ..., d-1\}$, $d \geq 2$:

$$\lim_{n \to \infty} n^{-(d-1)/(d+1)} \mathbb{E}\, f_\ell(K_n) = e_{\ell,d} \int_{\partial K} \kappa(x)^{1/(d+1)} dx.$$

- K is a convex polytope, $\ell \in \{0, 1, ..., d-1\}$, $d \geq 2$:

$$\lim_{n \to \infty} (\log n)^{-(d-1)} \mathbb{E}\, f_\ell(K_n) = e'_{\ell,d} \cdot \text{number of flags of K}.$$

(flag is a maximal chain of faces, each a sub-face of the next in the chain)

- K_n is convex hull of n i.i.d. standard normal r.v. on $\mathbb{R}^d$:

$$\lim_{n \to \infty} (\sqrt{\log n})^{-(d-1)} \mathbb{E}\, f_\ell(K_n) = E_{\ell,d}.$$

Central limit theorems

Assume K has either a C^2 boundary or is a convex polytope.

If the random variable Z_n is either $\mathrm{Vol}_d(K_n)$ or $f_\ell(K_n)$, $\ell \in \{0, ..., d-1\}$, then

$$\sup_{x \in \mathbb{R}} \left| P\left[\frac{Z_n - \mathbb{E}\, Z_n}{\sqrt{\mathrm{Var} Z_n}} \le x \right] - \Phi(x) \right| \le c(K)\epsilon(n) = o(1).$$

$d = 2$, $f_0(K_n), K$ polygon: Groeneboom ('88); Cabo+Groeneboom ('94).

Central limit theorems

Assume K has either a C^2 boundary or is a convex polytope.

If the random variable Z_n is either $\mathrm{Vol}_d(K_n)$ or $f_\ell(K_n)$, $\ell \in \{0, ..., d-1\}$, then

$$\sup_{x \in \mathbb{R}} \left| P\left[\frac{Z_n - \mathbb{E}\, Z_n}{\sqrt{\mathrm{Var} Z_n}} \le x \right] - \Phi(x) \right| \le c(K)\epsilon(n) = o(1).$$

$d = 2$, $f_0(K_n), K$ polygon: Groeneboom ('88); Cabo+Groeneboom ('94).
$d = 2$, $K = B^2$, $\mathrm{Vol}_2(K_n)$: Hsing ('94).

Central limit theorems

Assume K has either a C^2 boundary or is a convex polytope.

If the random variable Z_n is either $\mathrm{Vol}_d(K_n)$ or $f_\ell(K_n)$, $\ell \in \{0, ..., d-1\}$, then

$$\sup_{x\in\mathbb{R}} \left| P\left[\frac{Z_n - \mathbb{E}\, Z_n}{\sqrt{\mathrm{Var} Z_n}} \leq x \right] - \Phi(x) \right| \leq c(K)\epsilon(n) = o(1).$$

$d = 2$, $f_0(K_n), K$ polygon: Groeneboom ('88); Cabo+Groeneboom ('94).
$d = 2$, $K = B^2$, $\mathrm{Vol}_2(K_n)$: Hsing ('94).

General smooth sets in $\mathbb{R}^d$: Reitzner ('05) and Vu ('06)

Central limit theorems

Assume K has either a C^2 boundary or is a convex polytope.

If the random variable Z_n is either $\mathrm{Vol}_d(K_n)$ or $f_\ell(K_n)$, $\ell \in \{0, ..., d-1\}$, then

$$\sup_{x \in \mathbb{R}} \left| P\left[\frac{Z_n - \mathbb{E}\, Z_n}{\sqrt{\mathrm{Var} Z_n}} \le x \right] - \Phi(x) \right| \le c(K)\epsilon(n) = o(1).$$

$d = 2$, $f_0(K_n), K$ polygon: Groeneboom ('88); Cabo+Groeneboom ('94).
$d = 2$, $K = B^2$, $\mathrm{Vol}_2(K_n)$: Hsing ('94).

General smooth sets in $\mathbb{R}^d$: Reitzner ('05) and Vu ('06)

Gaussian polytopes: Bárány + Vu ('07).

Central limit theorems

Assume K has either a C^2 boundary or is a convex polytope.

If the random variable Z_n is either $\mathrm{Vol}_d(K_n)$ or $f_\ell(K_n)$, $\ell \in \{0, ..., d-1\}$, then

$$\sup_{x \in \mathbb{R}} \left| P\left[\frac{Z_n - \mathbb{E}\, Z_n}{\sqrt{\mathrm{Var} Z_n}} \leq x \right] - \Phi(x) \right| \leq c(K)\epsilon(n) = o(1).$$

$d = 2$, $f_0(K_n), K$ polygon: Groeneboom ('88); Cabo+Groeneboom ('94).
$d = 2$, $K = B^2$, $\mathrm{Vol}_2(K_n)$: Hsing ('94).

General smooth sets in $\mathbb{R}^d$: Reitzner ('05) and Vu ('06)

Gaussian polytopes: Bárány + Vu ('07).

Polytopes in $\mathbb{R}^d$: Bárány + Reitzner ('10).

Variance asymptotics

CLT uses dependency graphs and requires lower bounds on variances.

Variance asymptotics

CLT uses dependency graphs and requires lower bounds on variances.

If the random variable Z_n is either $\mathrm{Vol}_d(K_n)$ or $f_\ell(K_n)$, $\ell \in \{0, ..., d-1\}$, then what is $\lim_{n\to\infty} \mathrm{Var} Z_n$?

This question arose 20 years ago.

Variance asymptotics

CLT uses dependency graphs and requires lower bounds on variances.

If the random variable Z_n is either $\mathrm{Vol}_d(K_n)$ or $f_\ell(K_n)$, $\ell \in \{0, ..., d-1\}$, then what is $\lim_{n\to\infty} \mathrm{Var} Z_n$?

This question arose 20 years ago.

Solution had been known only when K is unit disc or polytope in $\mathbb{R}^2$ (Groeneboom, Cabo + Groeneboom).

Variance asymptotics

- ∂K of class C^3, $\ell \in \{0, 1, ..., d-1\}$, $d \geq 2$:

$$\lim_{n\to\infty} n^{-(d-1)/(d+1)} \mathrm{Var} f_\ell(K_n) = (\mathrm{Vol} K)^{-(d-1)/(d+1)} \int_{\partial K} \kappa(x)^{1/(d+1)} dx \cdot V_{\ell,d}.$$

Variance asymptotics

- ∂K of class C^3, $\ell \in \{0, 1, ..., d-1\}$, $d \geq 2$:

$$\lim_{n\to\infty} n^{-(d-1)/(d+1)} \mathrm{Var} f_\ell(K_n) = (\mathrm{Vol} K)^{-(d-1)/(d+1)} \int_{\partial K} \kappa(x)^{1/(d+1)} dx \cdot V_{\ell,d}.$$

- K_n is Gaussian polytope, $\ell \in \{0, 1, ..., d-1\}$, $d \geq 2$:

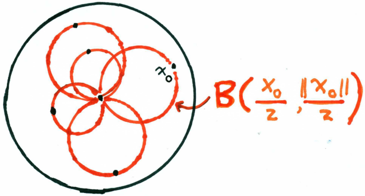
$$\lim_{n\to\infty} (2\log n)^{-(d-1)/2} \mathrm{Var} f_\ell(K_n) = v_{\ell,d}.$$

Calka, Schreiber and Y ('13), Calka and Y ('13,'14)

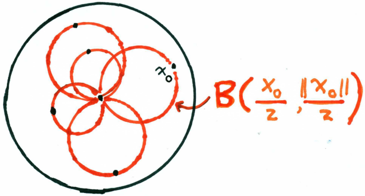
Scaling limits of convex hulls

What is the scaling limit of the boundary of the convex hull?

Scaling limits of convex hulls in unit ball

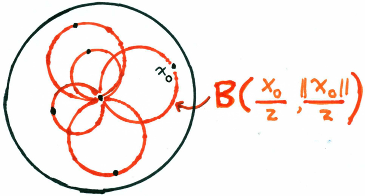


Scaling limits of convex hulls in unit ball



(i) x_0 is extreme in $\mathcal{X}$ iff $B(x_0/2, |x_0|/2)$ is not covered by $\bigcup_{x \in \mathcal{X}: x \neq x_0} B(x/2, |x|/2)$.

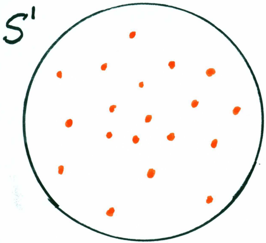
Scaling limits of convex hulls in unit ball



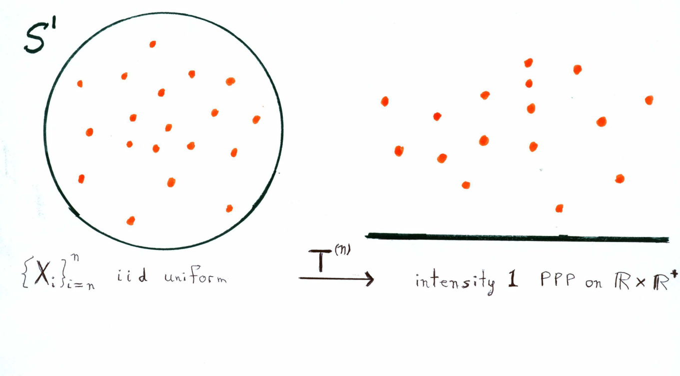
(i) x_0 is extreme in $\mathcal{X}$ iff $B(x_0/2, |x_0|/2)$ is not covered by $\bigcup_{x \in \mathcal{X}: x \neq x_0} B(x/2, |x|/2)$.

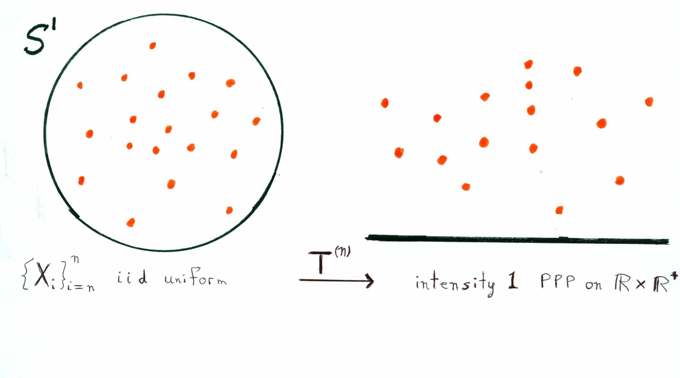
(ii) Scaling limit should preserve this property. Near x_0, balls have locally parabolic boundaries wrt polar coordinates; thus any reasonable scaling should have the property that its scaling in radial direction should be square of scaling in angular direction.

Scaling limits of convex hulls in unit ball



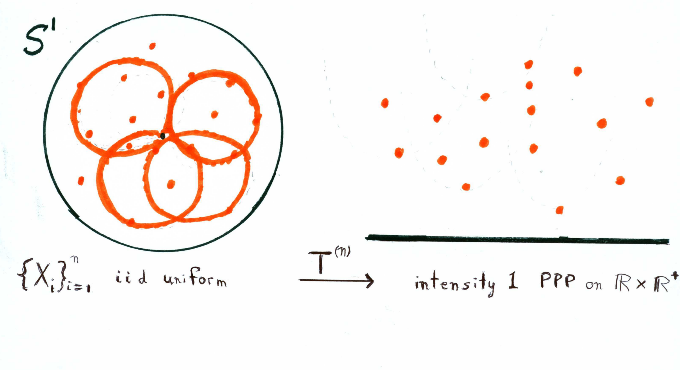
Scaling limits of convex hulls in unit ball



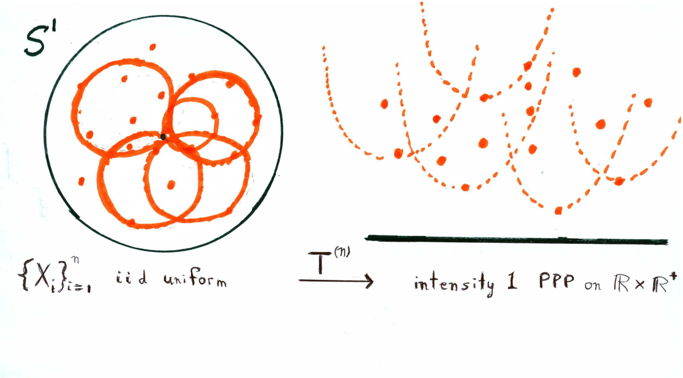


Fact: Scaling limit of $\{X_i\}_{i=1}^n$ under $T^{(n)}, n \to \infty$, is rate 1 PPP on $\mathbb{R} \times \mathbb{R}^+$.

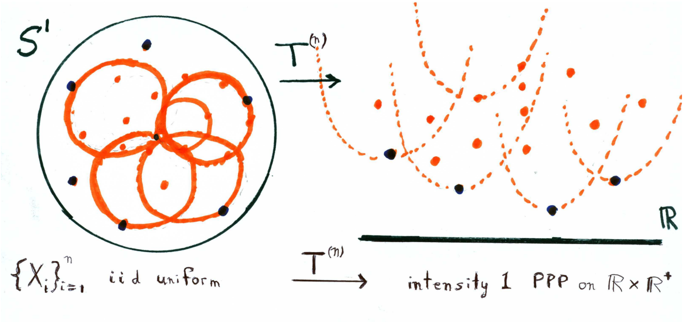
Scaling limits of convex hulls in unit ball



Scaling limits of convex hulls in unit ball

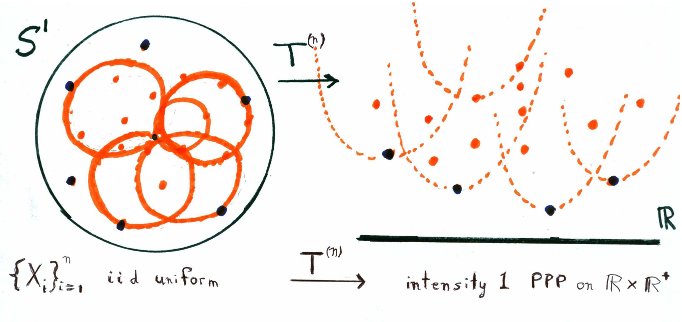


Scaling limits of convex hulls in unit ball



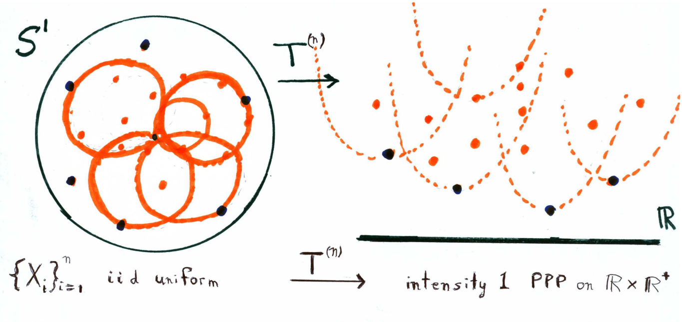
'Characterizing balls' are mapped to 'characterizing parabolas'.

Scaling limits of convex hulls in unit ball



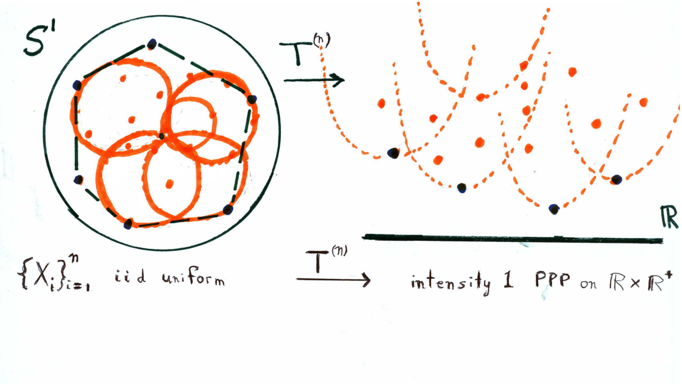
'Characterizing balls' are mapped to 'characterizing parabolas'.
Scaling limit of extreme points = thinned rate 1 PPP.

Scaling limits of convex hulls in unit ball

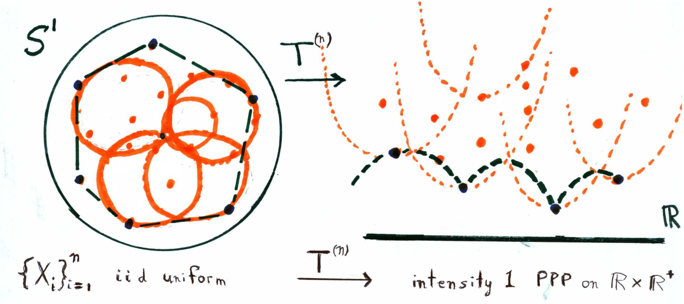


'Characterizing balls' are mapped to 'characterizing parabolas'.
Scaling limit of extreme points = thinned rate 1 PPP.
What about scaling limit of boundary?

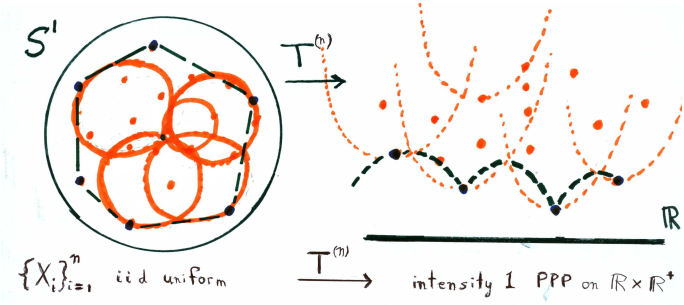
Scaling limits of convex hulls in unit ball



Scaling limits of convex hulls in unit ball



Scaling limits of convex hulls in unit ball



Thm: The scaling limit of $T^{(n)}(\partial K_n), n \to \infty$, is the (Burgers') festoon of parabolic surfaces (green) (Calka, Schreiber, Y.)

Scaling limits of convex hulls: scaling transform $T^{(n)}$

T_{u_0}: tangent space to $\mathbb{S}^{d-1}$ at $u_0 = (0, 0, ..., 1)$.

Exponential map $\exp : T_{u_0} \to \mathbb{S}^{d-1}$ maps a vector $v \in T_{u_0}$ to the point $u \in \mathbb{S}^{d-1}$ lying at the end of the geodesic of length $|v|$ starting at u_0 and having direction v.

Scaling limits of convex hulls: scaling transform $T^{(n)}$

T_{u_0}: tangent space to $\mathbb{S}^{d-1}$ at $u_0 = (0, 0, ..., 1)$.

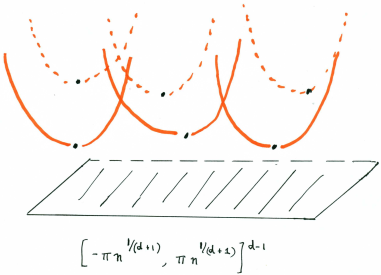
Exponential map $\exp : T_{u_0} \to \mathbb{S}^{d-1}$ maps a vector $v \in T_{u_0}$ to the point $u \in \mathbb{S}^{d-1}$ lying at the end of the geodesic of length $|v|$ starting at u_0 and having direction v.

Scaling transform $T^{(n)} : B^d \mapsto \mathbb{R}^{d-1} \times \mathbb{R}$

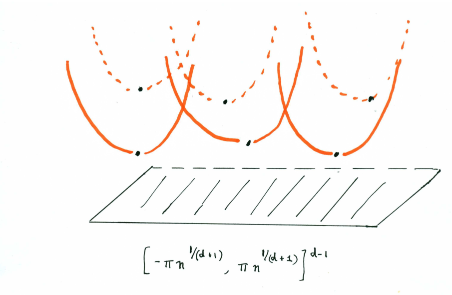
$$T^{(n)}(x) := \left(n^{1/(d+1)} \exp^{-1}(\frac{x}{|x|}), n^{2/(d+1)}(1 - |x|) \right), \ x \in B^d \setminus \{\mathbf{0}\}.$$

The previous pictures showed what happens in the limit as $n \to \infty$. For fixed n the re-scaled picture looks like this:

Scaling limits of convex hulls in unit ball



Scaling limits of convex hulls in unit ball

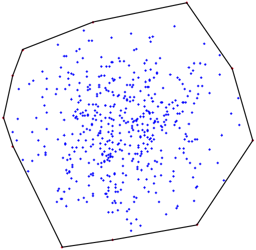


$K_n = \text{conv}(\{X_i\}_{i=1}^n)$. Then $\mathbb{E}\, f_0(K_n) =$

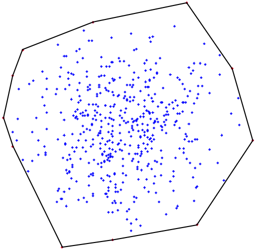
$$= \mathbb{E}\,(\text{card. extreme pts in } [-n^{1/(d+1)}, n^{1/(d+1)}]^{d-1} \times [0, n^{2/(d+1)}])$$

$\sim n^{(d-1)/(d+1)}$.

Scaling limits for Gaussian polytopes



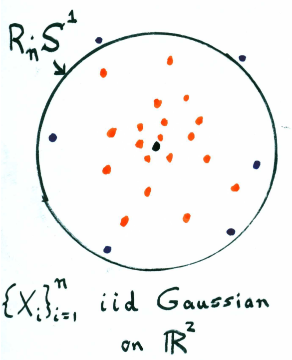
Scaling limits for Gaussian polytopes



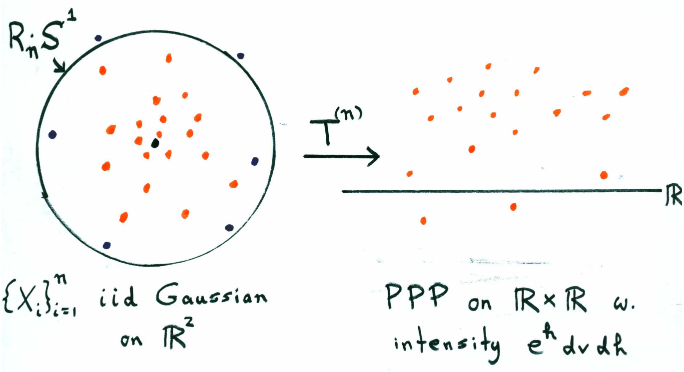
Extreme points are 'distant R_n from origin',

$$R_n := \sqrt{2 \log n - \log(2 \cdot (2\pi)^d \cdot \log n)}.$$

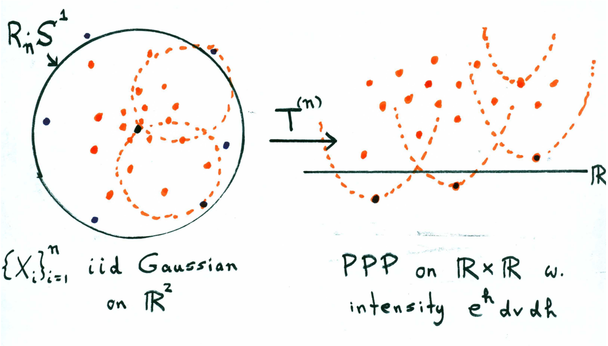
Scaling limits for Gaussian polytopes



Scaling limits for Gaussian polytopes

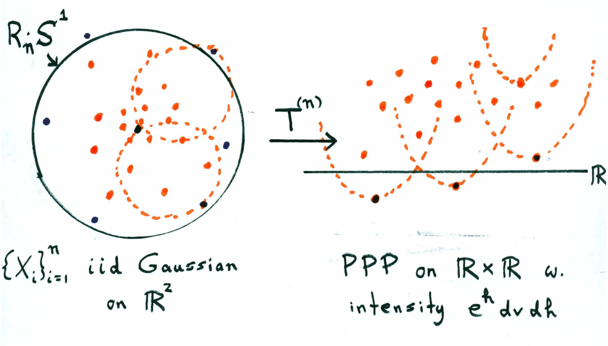


Fact: Scaling limit of $\{X_i\}_{i=1}^n$ is PPP on $\mathbb{R}^{d-1} \times \mathbb{R}$ with intensity $e^h dvdh$.



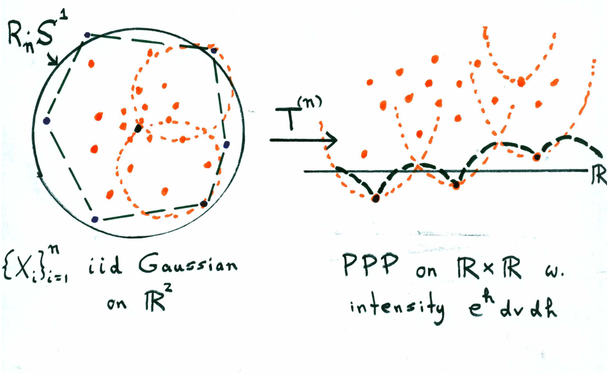
$T^{(n)}$ maps 'characterizing balls' to 'characterizing parabolas'.

Scaling limits for Gaussian polytopes

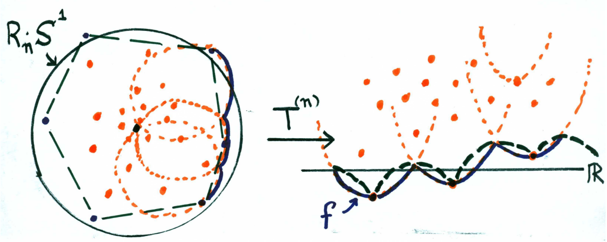


$T^{(n)}$ maps 'characterizing balls' to 'characterizing parabolas'.
Scaling limit of extreme points = thinned non-homogenous PPP on $\mathbb{R}^{d-1} \times \mathbb{R}$.

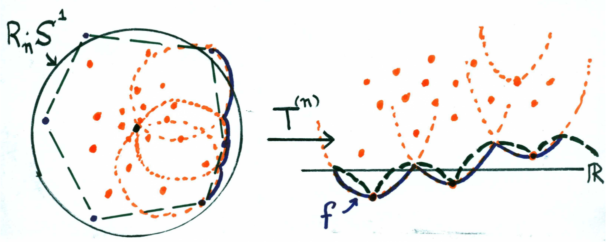
Scaling limits for Gaussian polytopes



Scaling limits for Gaussian polytopes

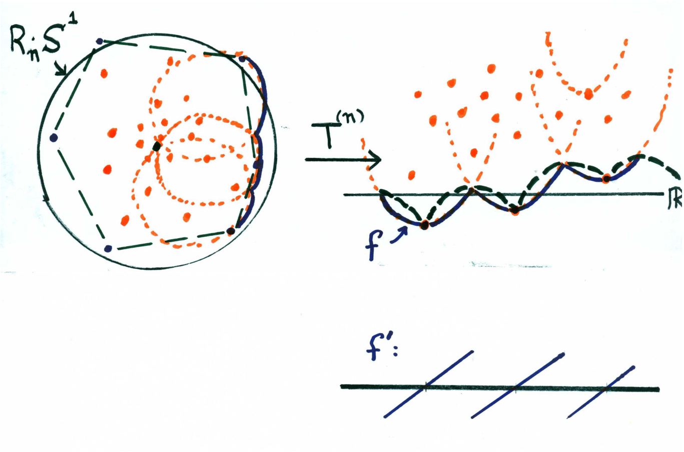


Scaling limits for Gaussian polytopes



Thm: The scaling limit of $T^{(n)}(\partial K_n), n \to \infty$, is the Burgers' festoon of parabolic surfaces touching points in PPP on $\mathbb{R}^{d-1} \times \mathbb{R}$ with intensity $d\mathcal{P}((v,h)) = e^h dh dv$. (Calka, Y.)

Scaling limits for Gaussian polytopes



Thm: The graph of the derivative of support function of convex hull converges after re-scaling to saw-tooth function f'. (Calka, Y.)

Scaling limits for Gaussian polytopes

$$R_n := \sqrt{2 \log n - \log(2 \cdot (2\pi)^d \cdot \log n)}.$$

Scaling limits for Gaussian polytopes

$$R_n := \sqrt{2\log n - \log(2\cdot(2\pi)^d \cdot \log n)}.$$

Define scaling transform $T^{(n)} : \mathbb{R}^d \to \mathbb{R}^{d-1} \times \mathbb{R}$

$$T^{(n)}(x) := \left(R_n \exp^{-1} \frac{x}{|x|},\ R_n^2 (1 - \frac{|x|}{R_n}) \right), \quad x \in \mathbb{R}^d \setminus \mathbf{0}.$$

Scaling limits for Gaussian polytopes

$R_n := \sqrt{2 \log n - \log(2 \cdot (2\pi)^d \cdot \log n)}.$

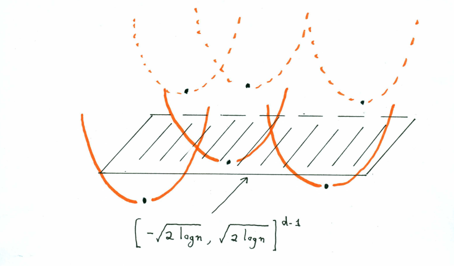
Define scaling transform $T^{(n)} : \mathbb{R}^d \to \mathbb{R}^{d-1} \times \mathbb{R}$

$$T^{(n)}(x) := \left(R_n \exp^{-1} \frac{x}{|x|}, \; R_n^2 (1 - \frac{|x|}{R_n}) \right), \quad x \in \mathbb{R}^d \setminus \mathbf{0}.$$

The transform $T^{(n)}$ does the job shown on previous slides.

What happens for fixed n?

Scaling limits for Gaussian polytopes

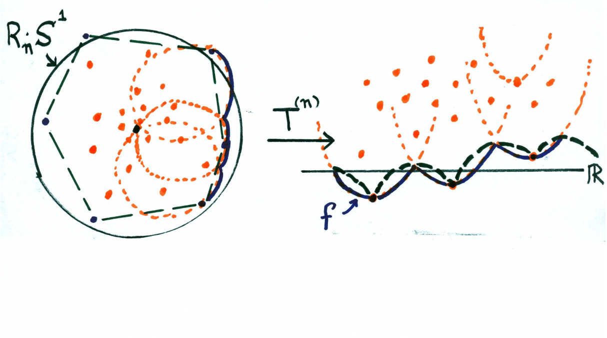


Variance asymptotics (Gaussian polytopes)

K_n is Gaussian polytope, $\ell \in \{0, 1, ..., d\}$:

$$\lim_{n\to\infty} (2\log n)^{-(d-1)/2} \mathrm{Var} f_\ell(K_n) = v_{\ell,d}.$$

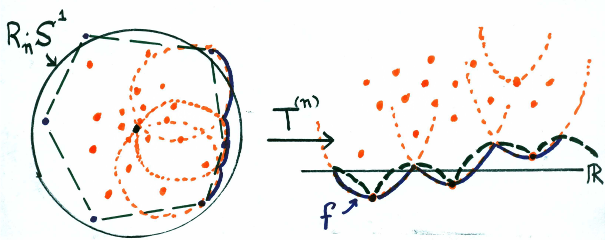
Formula for $v_{\ell,d}$?



$\mathcal{P}$: PPP on $\mathbb{R}^{d-1} \times \mathbb{R}$ with intensity $e^h dhdv$

$$\xi(x, \mathcal{P}) := \begin{cases} 1 \text{ if } x \oplus \Pi^{\uparrow} \text{ not covered by } \bigcup_{y \in \mathcal{P}, y \neq x} y \oplus \Pi^{\uparrow} \\ 0 \text{ otherwise.} \end{cases}$$

Variance asymptotics (Gaussian polytopes)



$\mathcal{P}$: PPP on $\mathbb{R}^{d-1} \times \mathbb{R}$ with intensity $e^h dh dv$

$$\xi(x, \mathcal{P}) := \begin{cases} 1 \text{ if } x \oplus \Pi^{\uparrow} \text{ not covered by } \bigcup_{y \in \mathcal{P}, y \neq x} y \oplus \Pi^{\uparrow} \\ 0 \text{ otherwise.} \end{cases}$$

Fact: ξ stabilizes.

Variance asymptotics (Gaussian polytopes)

For all $w_1, w_2 \in \mathbb{R}^d$ put

$$c^\xi(w_1, w_2) :=$$

$$\mathbb{E}\, \xi(w_1, \mathcal{P} \cup \{w_2\})\xi(w_2, \mathcal{P} \cup \{w_1\}) - \mathbb{E}\, \xi(w_1, \mathcal{P})\mathbb{E}\, \xi(w_2, \mathcal{P})$$

and

$$V_{0,d} := \int_{-\infty}^{\infty} \mathbb{E}\, \xi((\mathbf{0}, h), \mathcal{P}) dh$$

$$+ \int_{-\infty}^{\infty} \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{\infty} c^\xi((\mathbf{0}, h), (v, h'))e^{h'} e^h dh' dv dh.$$

Variance asymptotics (Gaussian polytopes)

For all $w_1, w_2 \in \mathbb{R}^d$ put

$$c^\xi(w_1, w_2) :=$$

$$\mathbb{E}\,\xi(w_1, \mathcal{P} \cup \{w_2\})\xi(w_2, \mathcal{P} \cup \{w_1\}) - \mathbb{E}\,\xi(w_1, \mathcal{P})\mathbb{E}\,\xi(w_2, \mathcal{P})$$

and

$$V_{0,d} := \int_{-\infty}^{\infty} \mathbb{E}\,\xi((\mathbf{0}, h), \mathcal{P}) dh$$

$$+ \int_{-\infty}^{\infty} \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{\infty} c^\xi((\mathbf{0}, h), (v, h')) e^{h'} e^h dh' dv dh.$$

Then

$$\lim_{n \to \infty} (2 \log n)^{-(d-1)/2} \mathrm{Var} f_0(K_n) = d\kappa_d V_{0,d}.$$

Inviscid Burgers' equation; $u_t + uu_x = 0$; initial condition specified by η

$u(t,x) =$ velocity

Initial conditions specified by a mean zero stationary Gaussian process η having covariance

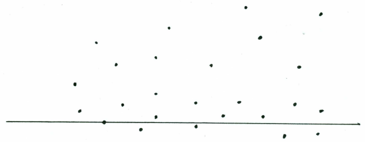
$$\mathbb{E}\,\eta(\mathbf{0})\eta(x) = o(1/\log x), x \to \infty$$

and

$$\mathbb{E}\,\eta(\mathbf{0})\eta(x) = 1 - a_2 x^2/2! + a_4 x^4/4! + o(x^4).$$

Inviscid Burgers' equation; $u_t + uu_x = 0$; initial condition specified by η

Let $\mathcal{P}$ be PPP on $\mathbb{R}^{d-1} \times \mathbb{R}$ with intensity $d\mathcal{P}((v,h)) = e^{-h} dv dh$.

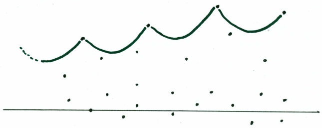


Inviscid Burgers' equation; $u_t + uu_x = 0$; initial condition specified by η

We 'thin' $\mathcal{P}$ using translates of $y = x^2/2$; the resulting point set gives a dependent thinning of $d\mathcal{P}((v,h)) = e^{-h} dv dh$.

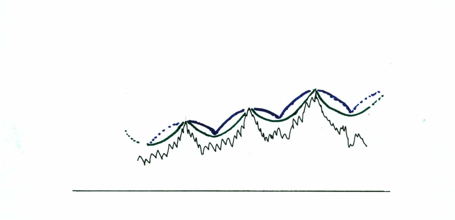
Inviscid Burgers' equation; $u_t + uu_x = 0$; initial condition specified by η

We 'thin' $\mathcal{P}$ using translates of $y = x^2/2$; the resulting point set gives a dependent thinning of $d\mathcal{P}((v,h)) = e^{-h} dv dh$.



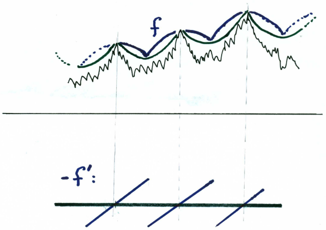
Inviscid Burgers' equation; $u_t + uu_x = 0$; initial condition specified by η

At each local max we put a translate of the inverted parabola $y = -x^2/2$.



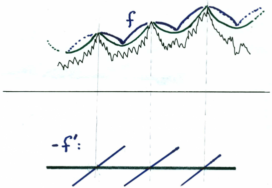
Consider the derivative of the inverted festoon of translates....

Inviscid Burgers' equation; $u_t + uu_x = 0$; initial condition specified by η



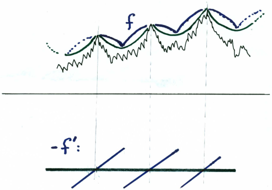
Fix t. The limit velocity process $u(L^2 t, L^2 x)$, subject to $L^2\sqrt{2\log L} \times \eta(x/L)$, converges as $L \to \infty$ to the sawtooth graph $-f'$ (Molchanov, Surgailis, Woyczynski, '95).

Inviscid Burgers' equation; $u_t + uu_x = 0$; initial condition specified by η



(i) local min of the green festoon $\leftrightarrow$ shocks in the limit velocity process $u(L^2t, L^2x)$, $L \to \infty$; (MSW, '95).

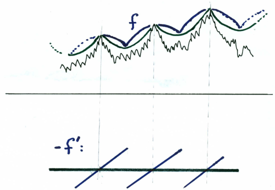
Inviscid Burgers' equation; $u_t + uu_x = 0$; initial condition specified by η



(i) local min of the green festoon $\leftrightarrow$ shocks in the limit velocity process $u(L^2t, L^2x)$, $L \to \infty$; (MSW, '95).

(ii) local max of festoon $\leftrightarrow$ zeros of limit velocity process.

Inviscid Burgers' equation; $u_t + uu_x = 0$; initial condition specified by η



(i) local min of the green festoon $\leftrightarrow$ shocks in the limit velocity process $u(L^2 t, L^2 x)$, $L \to \infty$; (MSW, '95).

(ii) local max of festoon $\leftrightarrow$ zeros of limit velocity process.

(iii) re-scaled angular increments between consecutive extreme points in K_n behave like the spacings between zeros of the zero-viscosity solution.

Inviscid Burgers' equation $u_t + uu_x = 0$

The correspondence between extreme points of convex hulls of gaussian samples and zero-viscosity solutions to Burgers' equation merits further investigation.

Are some aspects of the convex hull geometry captured by a stochastic PDE?

THANK YOU