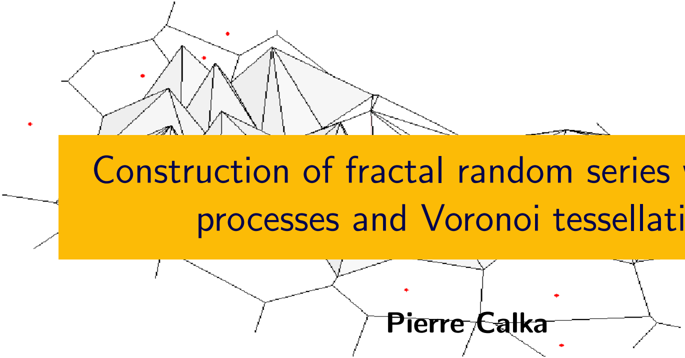


Construction of fractal random series with point processes and Voronoi tessellations

Pierre Calka

Simons Workshop on Stochastic Geometry and Point Processes

University of Texas in Austin, *8 May 2014*

Outline

Weierstrass and Takagi functions

A new Takagi-type series with a Poisson-Voronoi basis

Elements of proof

A dual model

Joint work with **Yann Demichel** (Université Paris Ouest)

Outline

Weierstrass and Takagi functions

- The Weierstrass function (1872)
- Fractal properties of the Weierstrass function
- The Takagi function (1903)
- Fractal properties of T
- Generalization of the Takagi function
- Takagi function in dimension $D \geq 2$
- A toy model
- Randomization of the Takagi model

A new Takagi-type series with a Poisson-Voronoi basis

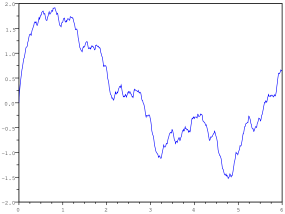
Elements of proof

A dual model

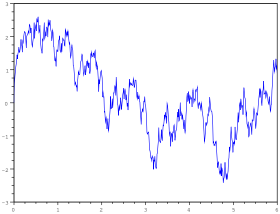
The Weierstrass function (1872)

$$W_{\lambda,\alpha}(x) = \sum_{n=0}^{\infty} \lambda^{-n\alpha} \sin(\lambda^n x), \quad x \in \mathbb{R}, \quad \lambda > 1,\ \alpha \in (0,1)$$

Property. The function $W_{\lambda,\alpha}$ is continuous, nowhere differentiable.



$\lambda = 1.5,\ \alpha = 0.9$



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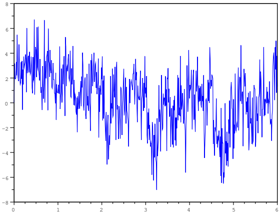


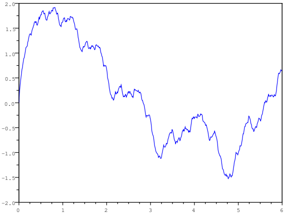
$\lambda = 1.5,\ \alpha = 0.1$

Fractal properties of the Weierstrass function

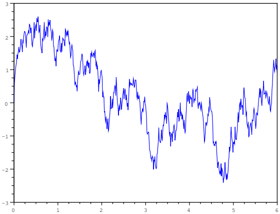
- Graph $\mathrm{Gr}(W_{\lambda,\alpha}) := \{(x, W_{\lambda,\alpha}(x)) : x \in (0,1)\}$
- $\mathcal{N}(\varepsilon) :=$ number of boxes from a ε-regular grid necessary to cover $\mathrm{Gr}(W_{\lambda,\alpha})$
- $\dim_B(\mathrm{Gr}(W_{\lambda,\alpha})) := \lim_{\varepsilon \to 0} \frac{\log \mathcal{N}(\varepsilon)}{-\log \varepsilon}$

Box dimension calculation.

$$\dim_B(\mathrm{Gr}(W_{\lambda,\alpha})) = 2 - \alpha$$



$\lambda = 1.5,\ \alpha = 0.9$



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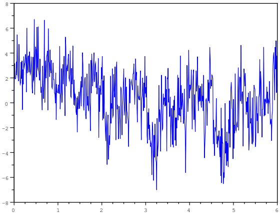


$\lambda = 1.5,\ \alpha = 0.1$

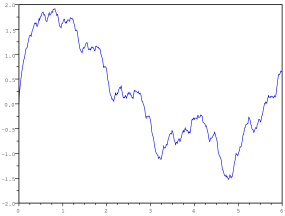
Fractal properties of the Weierstrass function

- s-dimensional Hausdorff measure

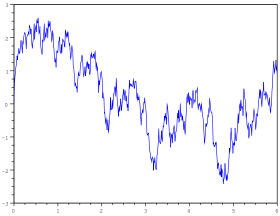
$$\mathcal{H}^s(\mathrm{Gr}(W_{\lambda,\alpha})) := \lim_{\varepsilon \to 0} \inf\{\sum_i \mathrm{diam}(U_i)^s : \mathrm{Gr}(W_{\lambda,\alpha}) \subset \cup_i U_i, 0 \le \mathrm{diam}(U_i) \le \varepsilon\}$$

- $\dim_H(\mathrm{Gr}(W_{\lambda,\alpha})) := \inf\{s > 0 : \mathcal{H}^s(\mathrm{Gr}(W_{\lambda,\alpha})) = 0\}$

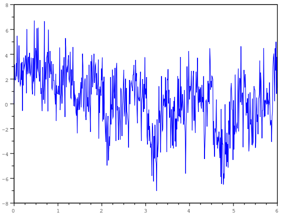
Hausdorff dimension calculation. Still open!



$\lambda = 1.5,\ \alpha = 0.9$



$\lambda = 1.5,\ \alpha = 0.5$

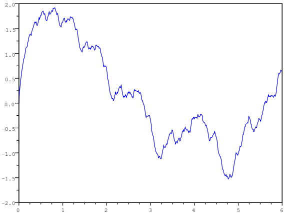


$\lambda = 1.5,\ \alpha = 0.1$

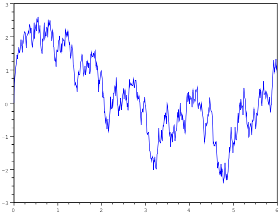
Regularity of the Weierstrass function

Property. The function $W_{\lambda,\alpha}$ is

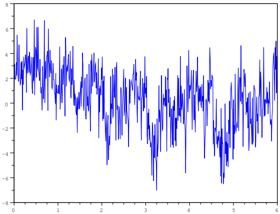
- α-Hölder for $\alpha \in (0,1)$,
- β-Hölder for every $\beta < 1$ if $\alpha = 1$,
- $\mathcal{C}^1$ for $\alpha > 1$.



$\lambda = 1.5,\ \alpha = 0.9$



$\lambda = 1.5,\ \alpha = 0.5$

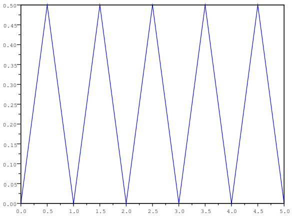


$\lambda = 1.5,\ \alpha = 0.1$

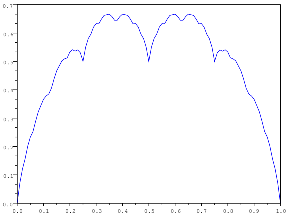
The Takagi function (1903)

$$T(x) = \sum_{n=0}^{\infty} 2^{-n} d(2^n x, \mathbb{Z}), \quad x \in \mathbb{R}$$

Property. The function T is continuous, α-Hölder for any $\alpha < 1$ and nowhere differentiable.



First layer of the sum

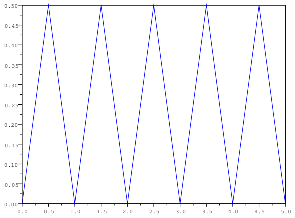


Graph of T

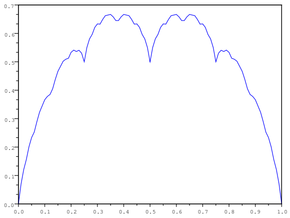
Fractal properties of T

Theorem. (Mauldin & Williams, 1986) $\dim_H(\mathrm{Gr}(T)) = 1$.

Remark. Counter-example to Marcinkiewicz's result for a function α-Hölder $\forall \alpha < 1$



First layer of the sum



Graph of T

Generalization of the Takagi function

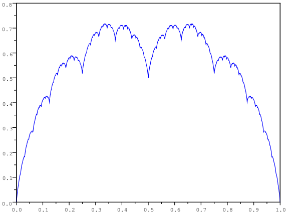
$$T_\alpha(x) = \sum_{n=0}^{\infty} 2^{-n\alpha} d(2^n x, \mathbb{Z}), \quad x \in \mathbb{R}, \quad \alpha \in (0,1)$$

Box dimension calculation.

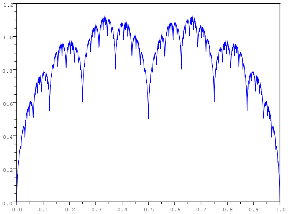
$$\dim_B(\mathrm{Gr}(T_\alpha)) = 2 - \alpha$$

Hausdorff dimension calculation. (Ledrappier, 1992)

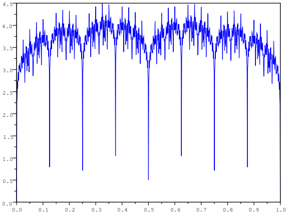
$$\dim_H(\mathrm{Gr}(T_\alpha)) = 2 - \alpha \quad \text{for almost all } \alpha \in (0,1)$$



$\alpha = 0.9$



$\alpha = 0.5$



$\alpha = 0.1$

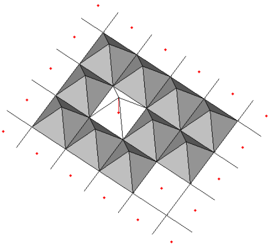
Takagi function in dimension $D \geq 2$

Each layer is a sawtooth or pyramidal function with $\mathbb{Z}^D$-basis.

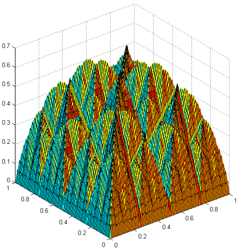
Box dimension calculation.

$$\dim_B(\mathrm{Gr}(T_\alpha)) = D + 1 - \alpha$$

Hausdorff dimension calculation. Still open!



One layer of the sum



Graph of T_α

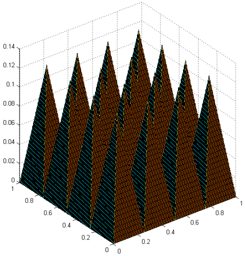
Takagi function in dimension $D \geq 2$

Each layer is a sawtooth or pyramidal function with $\mathbb{Z}^D$-basis.

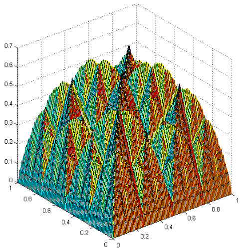
Box dimension calculation.

$$\dim_B(\mathrm{Gr}(T_\alpha)) = D + 1 - \alpha$$

Hausdorff dimension calculation. Still open!

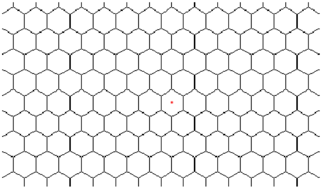


One layer of the sum



Graph of T_α

A toy model: the Takagi function over an hexagonal tiling of $\mathbb{R}^2$

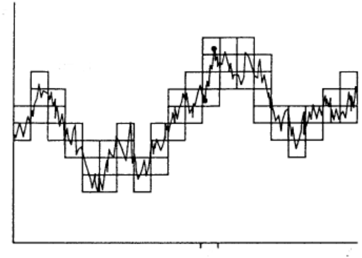


- Hexagonal tiling with 0 as a center and hexagons of diameter 2
- Function Δ: pyramidal function equal to 1 at any center of an hexagon and 0 on the grid

$$T_\alpha^{(\text{Hex})}(x) := \sum_{n=0}^{\infty} 2^{-n\alpha} \Delta(2^n x), \quad x \in \mathbb{R}^2, \alpha \in (0,1]$$

Theorem. $\dim_B(\text{Gr}(T_\alpha^{(\text{Hex})})) = 3 - \alpha$

Box dimension and oscillation estimates



$$\mathrm{Osc}_\varepsilon(x) := \sup_{y,y' \in x+[0,\varepsilon]^2} |T_\alpha^{(\mathrm{Hex})}(y) - T_\alpha^{(\mathrm{Hex})}(y')|, \quad x \in \mathbb{R}^2, \varepsilon > 0$$

Property. (K. Falconer)

$$\varepsilon^{-1} \sum_{k_1,k_2=0}^{\lfloor \varepsilon^{-1} \rfloor} \mathrm{Osc}_\varepsilon((\varepsilon k_1, \varepsilon k_2)) \leq \mathcal{N}(\varepsilon) \leq 2\lceil \varepsilon^{-1} \rceil^2 + \varepsilon^{-1} \sum_{k_1,k_2=0}^{\lfloor \varepsilon^{-1} \rfloor} \mathrm{Osc}_\varepsilon((\varepsilon k_1, \varepsilon k_2))$$

Box dimension of the toy model

- Use of the Hölder regularity:

$T_\alpha^{(\mathrm{Hex})}$ is α-Hölder so $\dim_B(\mathrm{Gr}(T_\alpha^{(\mathrm{Hex})})) \leq 3 - \alpha$.

- Lower-bound of the oscillation:

on a hexagon $\mathcal{C}_c^N$ of generation $N \geq 1$ with center $c \in \mathbb{R}^2$ and vertices $c_1, \cdots, c_6$

$$
\begin{aligned}
\sup_{y,y' \in \mathcal{C}_c^N} & |T_\alpha^{(\mathrm{Hex})}(y) - T_\alpha^{(\mathrm{Hex})}(y')| \\
& \geq |T_\alpha^{(\mathrm{Hex})}(c_i) - T_\alpha^{(\mathrm{Hex})}(c)| \\
& \geq \frac{1}{2}(|T_\alpha^{(\mathrm{Hex})}(c_i) - T_\alpha^{(\mathrm{Hex})}(c)| + |T_\alpha^{(\mathrm{Hex})}(c_j) - T_\alpha^{(\mathrm{Hex})}(c)|)
\end{aligned}
$$

Box dimension of the toy model

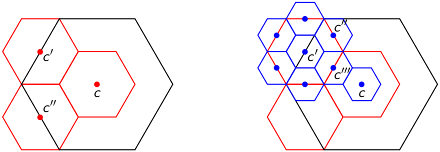
- Lower-bound of the oscillation:

$$\sup_{y,y' \in \mathcal{C}_c^N} |T_\alpha^{(\mathrm{Hex})}(y) - T_\alpha^{(\mathrm{Hex})}(y')| \geq C 2^{-N\alpha}$$

so

$$\dim_B(\mathrm{Gr}(T_\alpha^{(\mathrm{Hex})})) = \lim_{N \to \infty} \frac{\log(\mathcal{N}(2^{-N}))}{-\log(2^{-N})} \geq 3 - \alpha.$$

Requires the right choice of vertices c_i, c_j (depends on the *history* of the center c).



Randomization of the Takagi model

- Need of a more realistic model and easier to study

Motivation: rough surfaces...

- Possible randomizations

• Range of the layers:
construction of the Brownian bridge (Fournier, Fussell & Carpenter, 1982)

• Phase difference:
Hausdorff dimension of $W_{\lambda,\alpha}$ with random and independent translations of the layers (Hunt,1998)

• Basis of the pyramidal function

Outline

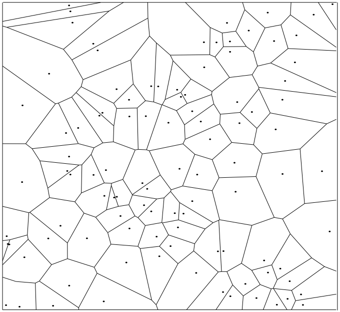
Weierstrass and Takagi functions

A new Takagi-type series with a Poisson-Voronoi basis
- The Voronoi tessellation
- Construction of the model
- Fractal properties of $F_{\lambda,\alpha,\beta}$

Elements of proof

A dual model

The Voronoi tessellation



- Euclidean space $\mathbb{R}^D$
- χ locally finite set of points
- For all $c \in \chi$,

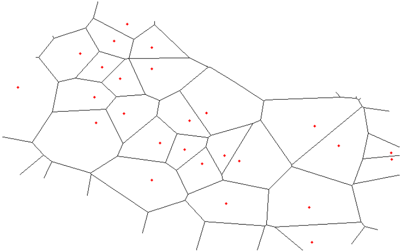
$$C(c|\chi) := \{y \in \mathbb{R}^D : \|y - c\| \leq \|y - c'\| \; \forall c' \in \chi\}$$

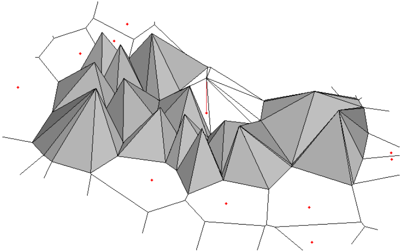
- Voronoi tessellation $\mathrm{Vor}(\chi)$: set of cells $C(c|\chi)$

Construction of the model

- $\{\chi_n, n \in \mathbb{N}\}$: sequence of independent homogeneous Poisson point processes with intensity one
- $\mathcal{M}_{n,\beta} := \lambda^{-\frac{n\beta}{D}} \mathrm{Vor}(\chi_n)$, $\beta > 0$
- Function $\Delta_{n,\beta}$: pyramidal function equal to

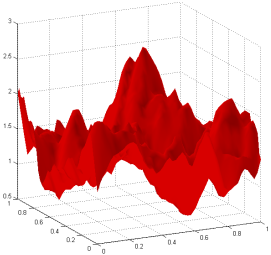
$$\begin{cases} 0 \text{ on the skeleton } \lambda^{-\frac{n\beta}{D}} \cup_{c \in \chi_n} \partial C(c|\chi_n) \\ 1 \text{ on the point process } \lambda^{-\frac{n\beta}{D}} \chi_n \end{cases}$$



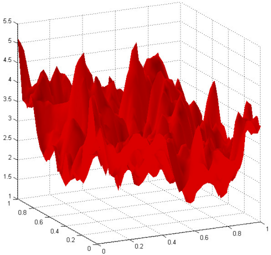


Construction of the model

$$F_{\lambda,\alpha,\beta}(x) = \sum_{n=0}^{\infty} \lambda^{-\frac{n\alpha}{D}} \Delta_{n,\beta}(x), \quad x \in \mathbb{R}, \quad \lambda > 1,\ 0 < \alpha \leq \beta < 1$$



$(\lambda, \alpha, \beta) = (1.5, 1, 1)$

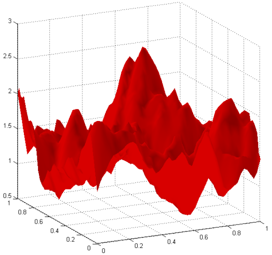


$(\lambda, \alpha, \beta) = (1.5, 0.2, 1)$

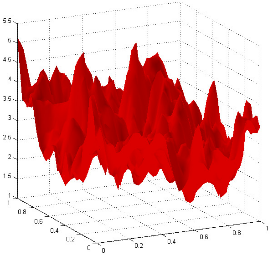
Fractal properties of $F_{\lambda,\alpha,\beta}$

Theorem. For any $0 < \alpha \leq \beta < 1$,

$$\dim_H(\mathrm{Gr}(F_{\lambda,\alpha,\beta})) = \dim_B(\mathrm{Gr}(F_{\lambda,\alpha,\beta})) = D + 1 - \frac{\alpha}{\beta} \quad \text{almost-surely.}$$



$(\lambda, \alpha, \beta) = (1.5, 1, 1)$



$(\lambda, \alpha, \beta) = (1.5, 0.2, 1)$

Outline

Weierstrass and Takagi functions

A new Takagi-type series with a Poisson-Voronoi basis

Elements of proof

- Strategy
- Use of Frostman's lemma
- Oscillation set
- Distribution of the height difference

A dual model

Strategy

- $\dim_H \leq \dim_B$: lower-bound for $\dim_H$, upper-bound for $\dim_B$
- Zero-one law: $\dim_B$ and $\dim_H$ almost-sure constants
- Upper-bound of $\dim_B$: use of the oscillation
- Lower-bound of $\dim_H$: use of Frostman's lemma, i.e. the finite energy criterion

Use of Frostman's lemma

Lemma.

If there exists a finite measure μ such that $\mu(\mathrm{Gr}(F_{\lambda,\alpha,\beta})) > 0$ and

$$I_s(\mu) = \iint_{(\mathbb{R}^{D+1})^2} \frac{1}{(|x-y|^2 + |h_x - h_y|^2)^{\frac{s}{2}}} \mathrm{d}\mu((x,h_x))\mathrm{d}\mu((y,h_y)) < \infty,$$

then $\dim_H(\mathrm{Gr}(F_{\lambda,\alpha,\beta})) \geq s$.

Initial idea. Take the image of $\mathrm{d}x$ by $x \longmapsto (x, F_{\lambda,\alpha,\beta}(x))$

Problem. Requires the distribution of $|F_{\lambda,\alpha,\beta}(x) - F_{\lambda,\alpha,\beta}(y)|$.
When is Δ_n linear in a vicinity of x and y?

Solution.
- Restrict $\mathrm{d}x$ to the set of points x s.t. Δ_n is linear in a vicinity of x
- Show that this set, called oscillation set, is large
- Estimate the height difference above this set

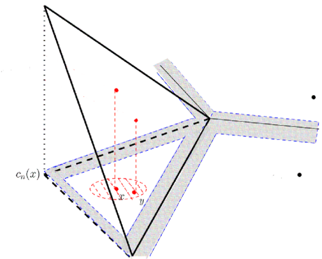
Oscillation set

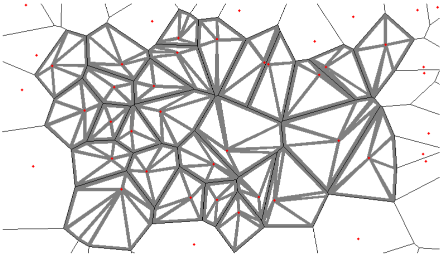
$H :> \beta$

$\mathcal{O}_n := \{x \in [0,1]^D : B(x, \lambda^{-\frac{nH}{D}})$ under the same face of a pyramid$\}$

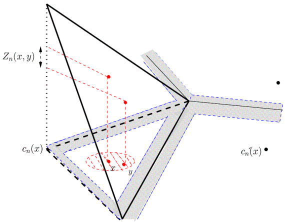
Proposition.

$$\mathbb{P}[x \notin \mathcal{O}_n] = O\left(\lambda^{\frac{n(\beta-H)}{D}}\right) \quad \text{and} \quad \lim_{N\to+\infty} \mathbb{P}[\mathrm{Vol}(\cap_{n\geq N}\mathcal{O}_n) > 0] = 1.$$





Distribution of the height difference



$c_n(x)$: nucleus of x
$c_n'(x)$: neighbor of $c_n(x)$ in direction $[c_n(x), x)$

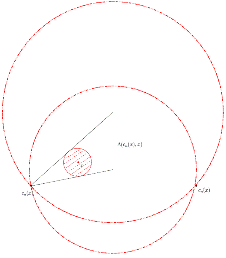
$Z_n(x,y) := \lambda^{-\frac{n\alpha}{D}}(\Delta_{n,\beta}(x) - \Delta_{n,\beta}(y))$
g_{Z_n}: density of Z_n conditional on $\{x \in \mathcal{O}_n\}$

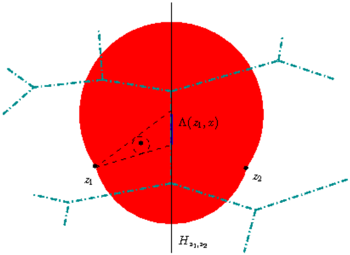
Proposition.

$$Z_n(x,y) = \frac{2\lambda^{\frac{n\alpha}{D}}}{\|c_n(x) - c_n'(x)\|^2}\langle x - y, c_n(x) - c_n'(x)\rangle,$$

$$\mathbb{P}[x \in \mathcal{O}_n]\|g_{Z_n}\|_\infty = O\left(\frac{\lambda^{-\frac{n(\beta-\alpha)}{D}}}{\|x-y\|}\right).$$

Distribution of the height difference: proof





Explicit density for the joint distribution of $(c_n(x), c_n'(x))$:

$$\varphi(z_1, z_2) = \exp\left(-\lambda^{n\beta}\mathrm{Vol}\left(\bigcup_{u\in\Lambda(z_1,x)} B(u, \|u - z_1\|)\right)\right)\mathbf{1}_{A_{n,x}}(z_1, z_2)$$

where $(z_1, z_2) \in A_{n,x}$ if $B(x, \lambda^{-\frac{nH}{D}})$ is *under the same pyramid face*.

Outline

Weierstrass and Takagi functions

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Elements of proof

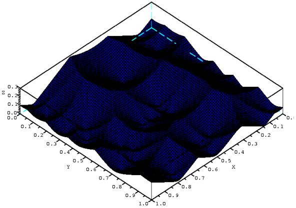
A dual model

- Construction of the dual model
- Convergence of the series
- Maximal distance to a Poisson point process

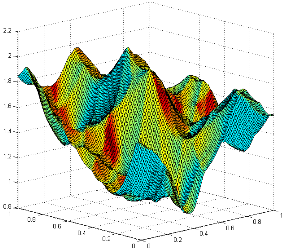
Construction of the dual model

$\{\chi_n, n \in \mathbb{N}\}$: sequence of independent homogeneous Poisson point processes with intensity one

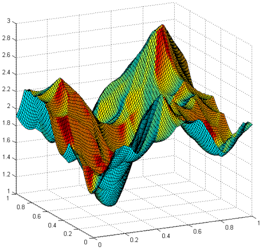
$$G_{\lambda,\alpha,\beta}(x) = \sum_{n=0}^{\infty} \lambda^{\frac{n\alpha}{D}} d(x, \lambda^{-\frac{n\beta}{D}} \chi_n), \quad x \in \mathbb{R}, \quad \lambda > 1, 0 < \alpha < \beta < 1$$



One layer of the sum



$(\lambda, \alpha, \beta) = (1.2, 0.1, 1)$

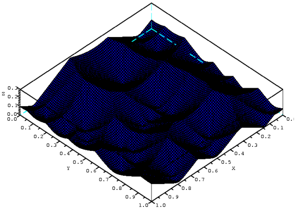


$(\lambda, \alpha, \beta) = (1.2, 0.5, 1)$

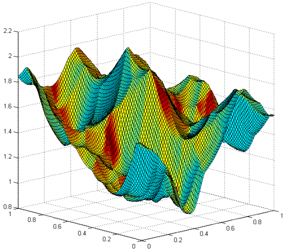
Construction of the dual model

By a similar method,

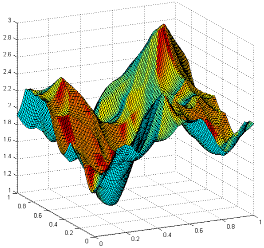
$$\dim_H(\mathrm{Gr}(G_{\lambda,\alpha,\beta})) = \dim_B(\mathrm{Gr}(G_{\lambda,\alpha,\beta})) = D + \frac{\alpha}{\beta} \quad \text{almost-surely.}$$



One layer of the sum



$(\lambda, \alpha, \beta) = (1.2, 0.1, 1)$



$(\lambda, \alpha, \beta) = (1.2, 0.5, 1)$

Convergence of the series

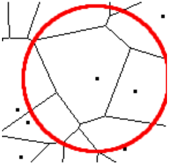
- Problem: the function $d(\cdot, \lambda^{-\frac{n\beta}{D}}\chi_n)$ is no longer uniformly bounded.

- Favorable event: when $[0,1]^D$ is divided into small equal cubes and each of these cubes intersects $\lambda^{\frac{n\beta}{D}}\chi_n$.

- Borel-Cantelli: it is enough to check for some $\gamma \in (\alpha, \beta)$ that

$$\sum_{n=0}^{\infty} \lambda^{n\gamma} \mathbb{P}[[0, \lambda^{\frac{n(\beta-\gamma)}{D}}]^D \cap \chi_0 = \emptyset] < \infty.$$

Maximal distance to a Poisson point process



$$R(c) := \inf\{r > 0 : B(c, r) \supset C(c|\lambda^{-\frac{n\beta}{D}}\chi_n)\},\ c \in \lambda^{-\frac{n\beta}{D}}\chi_n$$

$$\sup_{x\in[0,1]^D} d(x, \lambda^{-\frac{n\beta}{D}}\chi_n) = \sup_{c\in\lambda^{-\frac{n\beta}{D}}\chi_n} R(c) := R_{\max}(\lambda^{n\beta})$$

Theorem. (with Nicolas Chenavier)

$$\mathbb{P}[K_{1.D}\gamma R_{\max}(\gamma)^D - \log\left(K_{2,D}\gamma(\log\gamma)^{D-1}\right) \leq t] \underset{\gamma\to\infty}{\longrightarrow} e^{-e^{-t}}, \quad t \in \mathbb{R},$$

where $K_{1,D}$, $K_{2,D}$ are explicit positive constants.

Concluding remarks

- Estimates for other partitions which are random perturbations of the dyadic mesh
- Generalization to isotropic and stationary point processes
- Regularity of the series and multifractal study
- Similar models on a manifold

Thank you for your attention!