# Shift-Coupling and Mass-Stationarity

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# Mass-Stationarity

**Setting:** Let $(\Omega, \mathcal{F}, \mathbb{P})$ support the random elements below.

Let $G$ be a locally compact second countable topological group with left-invariant Haar measure $\lambda$.

Let $\xi$ be a random measure on $G$.

Let $X$ be a random element in a space on which $G$ acts.

Write $\theta_t$ for the shift map placing a new origin at $t \in G$.

E.g. $X = (X_s)_{s \in G}$ a shift-measurable r.f. and $\theta_t X = (X_{ts})_{s \in G}$.

**Definition**

The pair $(X, \xi)$ is called mass-stationary if for all bounded $\lambda$-continuity sets $C \subseteq G$ of positive $\lambda$-measure

$$\theta_{V_C}(X, \xi, U_C^{-1}) \stackrel{D}{=} (X, \xi, U_C^{-1}) \qquad \text{(randomised self-shift-coupling)}$$

where $U_C$ is such that $\mathbb{P}(U_C \in \cdot \mid X, \xi) = \lambda(\cdot \mid C)$

and $V_C$ is such that $\mathbb{P}(V_C \in \cdot \mid X, \xi, U_C) = \xi(\cdot \mid C)$.

# The case when $G$ is compact

Let $G$ be compact and $S$ be a random element in $G$ such that

$$\mathbb{P}(S \in \cdot \mid X, \xi) = \xi(\cdot \mid G).$$

Say that the origin is at a typical location for $X$ in the mass of $\xi$ if

$$\theta_S(X, \xi) \stackrel{D}{=} (X, \xi).$$

**Theorem**

Let $G$ be compact. Then $(X, \xi)$ is mass-stationary if (and only if) the origin is at a typical location for $X$ in the mass of $\xi$.

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**Theorem**

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**Recall the definition of mass-stationarity:**

The pair $(X, \xi)$ is called mass-stationary if for all bounded $\lambda$-continuity sets $C \subseteq G$ of positive $\lambda$-measure

$$\theta_{V_C}(X, \xi, U_C^{-1}) \stackrel{D}{=} (X, \xi, U_C^{-1}) \qquad \text{(randomised self-shift-coupling)}$$

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# Palm Theory

The pair $(X, \xi)$ is called a Palm version of a stationary $(\hat{X}, \hat{\xi})$ defined on some $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$, if for each nonnegative measurable function $f$ and each Borel subset $B$ of $G$ with $0 < \lambda(B) < \infty$,

$$\mathbb{E}[f(X, \xi)] = \hat{\mathbb{E}}\Big[\int_B f(\theta_t(\hat{X}, \hat{\xi}))\hat{\xi}(dt)\Big]\Big/\lambda(B).$$

Here $(X, \xi)$ and $(\hat{X}, \hat{\xi})$ are allowed to have distributions that are only $\sigma$-finite and not necessarily probability measures. The measure $\mathbb{P}$ is finite if and only if $\hat{\xi}$ has finite intensity.

**Theorem**

$(X, \xi)$ mass-stationary $\iff$ $(X, \xi)$ Palm version of a stationary pair

**Corollary**

$(X, \lambda)$ mass-stationary $\iff$ $X$ stationary

# Shift-Coupling and Total Variation Cesaro Limits

Let $\mathbb{P}$ and $\hat{\mathbb{P}}$ be probability measures such that probabilities of invariant sets are identical (always the case they are trivial).

Let $(X, \xi)$ be a (normalised) Palm version of a stationary $(\hat{X}, \hat{\xi})$.

**Theorem:** Let $G$ be general.

There exists (after extension) a random $T$ in $G$ such that

$$\theta_T(X, \xi) \stackrel{D}{=} (\hat{X}, \hat{\xi}) \qquad \text{(shift-coupling)}$$

Then with $0 < \lambda(B) < \infty$ and $\mathbb{P}(U_B \in \cdot \,|\, X, \xi) = \lambda(\cdot \,|\, B)$,

$$\|\mathbb{P}(\theta_{U_B}(X, \xi) \in \cdot) - \hat{\mathbb{P}}((\hat{X}, \hat{\xi}) \in \cdot)\| \leq \mathbb{E}[\lambda(B \cap TB)/\lambda(B)].$$

So if there exist Folner sets $B_t, t > 0$, (amenability) then

$$\mathbb{P}(\theta_{U_{B_t}}(X, \xi) \in \cdot) \xrightarrow{TV} \hat{\mathbb{P}}((\hat{X}, \hat{\xi}) \in \cdot), \quad t \to \infty.$$

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$$\mathbb{P}(\theta_{U_{B_t}}(X, \xi) \in \cdot) \xrightarrow{TV} \hat{\mathbb{P}}((\hat{X}, \hat{\xi}) \in \cdot), \quad t \to \infty,$$

and with $\hat{\mathbb{P}}(W_B \in \cdot \,|\, \hat{X}, \hat{\xi}) = \hat{\xi}(\cdot \,|\, B)$

$$\mathbb{P}(\theta_{U_{B_t}}(\hat{X}, \hat{\xi}) \in \cdot) \xrightarrow{TV} \hat{\mathbb{P}}((X, \xi) \in \cdot), \quad t \to \infty.$$

# Preserving shifts $\pi$ and allocation rules $\tau_\pi$

Let $\pi$ be a measurable map taking $\xi$ to a location $\pi(\xi)$ in $G$.
Define the induced (invariant) allocation rule $\tau_\pi = \tau_\pi^\xi$ by

$$\tau_\pi(s) = \pi(\theta_s \xi)s, \qquad s \in G.$$

Call $\pi$ preserving if $\tau_\pi$ preserves $\xi$, that is, if $\xi(\tau_\pi \in \cdot) = \xi$.

**Theorem**

$(X, \xi)$ mass-stationary and $\pi$ preserving $\Rightarrow$ $\theta_{\pi(\xi)}(X, \xi) \overset{D}{=} (X, \xi)$

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Note that if $\xi$ is a simple point process and $\pi(\xi) = 0$ when $\xi(\{0\}) = 0$ then $\pi$ is preserving if and only if $\tau$ is a bijection.

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**Theorem:** Let $G = \mathbb{R}$ and $\xi$ be diffuse. Then

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The $\pi_r(\xi) = \sup\{t \in \mathbb{R} : \ \xi([0, t]) = r\}$, $r \in \mathbb{R}$, are preserving.

The property

$$\forall \text{ preserving } \pi: \ \theta_{\pi(\xi)}(X, \xi) \stackrel{D}{=} (X, \xi)$$

is not sufficient to define mass-stationarity

# Stationary independent backgrounds $Y$

Let $Y$ be a random element in a space on which $G$ acts.
Call $Y$ a stationary independent background if $Y$ is stationary and independent of $(X, \xi)$.

Let $\pi$ be a measurable map taking $(Y, \xi)$ to $\pi(Y, \xi)$ in $G$.

Call $\pi$ preserving if $\tau_\pi$ preserves $\xi$, that is, if $\xi(\tau_\pi \in \cdot) = \xi$.

**Theorem:** Let $G = \mathbb{R}^d$ and $\xi$ be diffuse. Then

$(X, \xi)$ mass-stationary $\iff$ $\forall$ stationary independent backgrounds $Y$ and $\forall$ preserving $\pi$: $\theta_{\pi(Y,\xi)}(X, \xi) \overset{D}{=} (X, \xi)$

**Corollary:** Let $G = \mathbb{R}^d$. Then with $\lambda_1$ Lebesgue measure on $\mathbb{R}$

$(X, \xi)$ mass-stationary $\iff$ $\forall$ $Y$ that are stationary independent backgrounds for $(X, \xi \otimes \lambda_1)$ and $\forall$ preserving $\pi$:

$$\theta_{\pi(Y, \xi\otimes\lambda_1)}(X, \xi) \overset{D}{=} (X, \xi)$$

# Shift-Coupling

**Setting:** Below let $(H, \mathcal{H})$ be the space of $(X, \eta, \xi)$.

Let $\eta$ be another random measure on $G$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

Let $K$ be a $\sigma$-finite kernel from $(H, \mathcal{H}) \otimes (G, \mathcal{G})$ to $(G, \mathcal{G})$.

Let $K$ be invariant, that is, $K(\theta_t(\cdot, \cdot), \theta_t \cdot) = K$ for all $t \in G$.

Say that $K$ balances $\xi$ and $\eta$ if $\int K(((X, \eta, \xi), s), \cdot)\, \xi(ds) = \eta$.

Let $((X, \eta), \xi)$ be the Palm version of a stationary $((\hat{X}, \hat{\eta}), \hat{\xi})$.

**Theorem**

The Palm version of $((\hat{X}, \hat{\xi}), \hat{\eta})$ has the distribution

$$\mathbb{E}_{\mathbb{P}}\big[ \int 1_{\{\theta_s(X, \eta, \xi) \in \cdot\}}\, K(((X, \eta, \xi), 0), ds)\big]$$

if and only if $K$ balances $\xi$ and $\eta$ a.e. $\mathbb{P}$.

**Corollary:** Let $\pi$ take $(X, \eta, \xi)$ to $\pi(X, \eta, \xi)$ in $G$.

Then $\theta_{\pi(X, \eta, \xi)}((X, \eta), \xi)$ is the Palm version of $((\hat{X}, \hat{\xi}), \hat{\eta})$

if and only if $\xi(\tau_\pi \in \cdot) = \eta$ a.e. $\mathbb{P}$.

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**Corollary:** Let $\pi$ take $(X, \eta, \xi)$ to $\pi(X, \eta, \xi)$ in $G$.

Then $\theta_{\pi(X,\eta,\xi)}((X, \xi), \eta)$ is the Palm version of $((\hat{X}, \hat{\xi}), \hat{\eta})$
if and only if $\xi(\tau_\pi \in \cdot) = \eta$ a.e. $\mathbb{P}$.

**Theorem:** Let the intensities of $\hat{\xi}$ and $\hat{\eta}$ be positive and finite.

There exists a Markovian $K$ balancing $\xi$ and $\eta$ a.e. $\mathbb{P}$ if and only if, given the the invariant $\sigma$-algebra of $(\hat{X}, \hat{\eta}, \hat{\xi})$, the conditional intensities of $\hat{\xi}$ and $\hat{\eta}$ are identical a.e. $\hat{\mathbb{P}}$.

# The case $G = \mathbb{R}$ – Diffuse measures

**Theorem:** Let $G = \mathbb{R}$.

Let $\xi$ be diffuse and have infinite mass in both directions. Then for $r \in \mathbb{R}$, the $\pi_r$ defined by $\pi_r(\xi) = \sup\{t \in \mathbb{R} : \xi([0,t]) = r\}$ preserves $\xi$, that is, $\xi(\tau_{\pi_r} \in \cdot) = \xi$.

Thus if $(X, \xi)$ is mass-stationary $\theta_{\pi_r(\xi)}(X, \xi) \stackrel{D}{=} (X, \xi)$ for all $r \in \mathbb{R}$.

**Theorem:** Let $G = \mathbb{R}$.

Let $(\hat{\xi}, \hat{\eta})$ be a stationary pair of diffuse and singular random measures with infinite mass in both directions and with the same conditional intensity given their invariant $\sigma$-algebra. Then the $\pi$ defined by

$$\pi(\xi, \eta) = \inf\{t > 0 \colon \xi([0,t]) = \eta([0,t])\}$$

balances $\xi$ and $\eta$ a.e., that is, $\xi(\tau_\pi \in \cdot) = \nu$ a.e.

Thus if $((X, \eta), \xi)$ is the Palm version of a stationary $((\hat{X}, \hat{\eta}), \hat{\xi})$ then $\theta_{\pi(\xi)}((X, \xi), \eta)$ is the Palm version of $((\hat{X}, \hat{\xi}), \hat{\eta})$.

# Standard Brownian motion $B = (B_s)_{s \in \mathbb{R}}$

Let $B = (B_s)_{s \in \mathbb{R}}$ be the canonical two-sided standard Brownian motion. Let $\mathbb{P}_0$ be its distribution. In particular, $B_0 = 0$ a.e. $\mathbb{P}$. Let $\ell^x$ be local time at level $x \in \mathbb{R}$. With $\nu$ a probability measure, put $\ell^\nu = \int_\mathbb{R} \ell^x \nu(dx)$ and $\mathbb{P}_\nu = \int_\mathbb{R} \mathbb{P}(B + x \in \cdot)dx$.

Then $B$ is stationary under the $\sigma$-finite $\mathbb{P}_\lambda = \int_\mathbb{R} \mathbb{P}(B + x \in \cdot)dx$. And $(B, \ell^\nu)$ under $\mathbb{P}_\nu$ is the Palm version of $(B, \ell^\nu)$ under $\mathbb{P}_\lambda$. Thus with $T_r = \pi_r(\xi) = \sup\{t \in \mathbb{R} : \ell^0([0, t]) = r\}$

$$\text{under } \mathbb{P}_0: \quad \theta_{T_r} B \stackrel{D}{=} B, \quad r \in \mathbb{R}.$$

Also $\ell^0$ and $\ell^\nu$ under $\mathbb{P}_\lambda$ have the same conditional intensity given the (trivial) invariant $\sigma$-algebra. Thus with

$$T^\nu = \pi(\ell^0, \ell^\nu) = \inf\{t > 0 : \ell^0([0, t]) = \ell^\nu([0, t])\}$$

we have that

$$(B, \ell^\nu) \text{ under } \mathbb{P}_0 \text{ is the Palm version of } (B, \ell^\nu) \text{ under } \mathbb{P}_\lambda.$$