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Normal approximation of geometric Poisson functionals

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joint work with

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1. Chaos expansion of Poisson functionals

Setting

η is a Poisson process on some measurable space $(\mathbb{X}, \mathcal{X})$ with intensity measure λ. This is a random element in the space **N** of all integer-valued σ-finite measures on $\mathbb{X}$, equipped with the usual σ-field (and distribution Π_λ) with the following two properties

- The random variables $\eta(B_1), \dots, \eta(B_m)$ are stochastically independent whenever $B_1, \dots, B_m$ are measurable and pairwise disjoint.
- $$\mathbb{P}(\eta(B) = k) = \frac{\lambda(B)^k}{k!} \exp[-\lambda(B)], \quad k \in \mathbb{N}_0, B \in \mathcal{X},$$

  where $\infty^k e^{-\infty} := 0$ for all $k \in \mathbb{N}_0$.

Definition (Difference operator)

For a measurable function $f : \mathbf{N} \to \mathbb{R}$ and $x \in \mathbb{X}$ we define a function $D_x f : \mathbf{N} \to \mathbb{R}$ by

$$D_x f(\mu) := f(\mu + \delta_x) - f(\mu).$$

For $x_1, \dots, x_n \in \mathbb{X}$ we define $D^n_{x_1,\dots,x_n} f : \mathbf{N} \to \mathbb{R}$ inductively by

$$D^n_{x_1,\dots,x_n} f := D^1_{x_1} D^{n-1}_{x_2,\dots,x_n} f,$$

where $D^1 := D$ and $D^0 f = f$.

Lemma

For any $f \in L^2(\mathbb{P}_\eta)$

$$T_n f(x_1, \dots, x_n) := \mathbb{E} D^n_{x_1,\dots,x_n} f(\eta),$$

defines a function $T_n f \in L^2_s(\lambda^n)$.

Definition

Let $n \in \mathbb{N}$ and $g \in L^2(\lambda^n)$. Then $I_n(g)$ denotes the multiple Wiener-Itô integral of g w.r.t. the compensated Poisson process $\eta - \lambda$. For $c \in \mathbb{R}$ let $I_0(c) := c$. These integrals have the properties

$$\mathbb{E} I_n(g) I_n(h) = n! \langle \tilde{g}, \tilde{h} \rangle_n, \quad n \in \mathbb{N}_0,$$
$$\mathbb{E} I_m(g) I_n(h) = 0, \quad m \neq n.$$

Here

$$\tilde{g}(x_1, \dots, x_n) := \frac{1}{n!} \sum_{\pi \in \Sigma_n} g(x_{\pi(1)}, \dots, x_{\pi(n)})$$

denotes the symmetrization of g.

Definition

Let L^2_η denote the space of all random variables $F \in L^2(\mathbb{P})$ such that $F = f(\eta)$ $\mathbb{P}$-almost surely, for some measurable function (representative) $f : \mathbf{N} \to \mathbb{R}$.

Theorem (Wiener '38; Itô '56; Y. Ito '88; L. and Penrose '11)

For any $F \in L^2_\eta$ there are uniquely determined $f_n \in L^2_s(\lambda^n)$ such that $\mathbb{P}$-a.s.

$$F = \mathbb{E}F + \sum_{n=1}^{\infty} I_n(f_n).$$

Moreover, we have $f_n = \frac{1}{n!} T_n f$, where f is a representative of F.

2. Malliavin operators

Definition

Let $F \in L^2_\eta$ have representative f. Define $D_x F := D_x f(\eta)$ for $x \in \mathbb{X}$, and, more generally $D^n_{x_1,\dots,x_n} F := D^n_{x_1,\dots,x_n} f(\eta)$ for any $n \in \mathbb{N}$ and $x_1, \dots, x_n \in \mathbb{X}$. The mapping $(\omega, x_1, \dots, x_n) \mapsto D^n_{x_1,\dots,x_n} F(\omega)$ is denoted by $D^n F$ (or by DF in the case $n = 1$).

Theorem (Y. Ito '88; Nualart and Vives '90; L. and Penrose '11)

Suppose $F = \mathbb{E}F + \sum_{n=1}^{\infty} I_n(f_n) \in L^2_\eta$. *Then* $DF \in L^2(\mathbb{P} \otimes \lambda)$ *iff* F *is in* dom D *(the domain of the Malliavin difference operator), that is*

$$\sum_{n=1}^{\infty} nn! \int f_n^2 \, d\lambda^n < \infty.$$

In this case we have $\mathbb{P}$*-a.s. and for* λ*-a.e.* $x \in \mathbb{X}$ *that*

$$D_x F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(x, \cdot)).$$

Definition

Let $H \in L^2_\eta(\mathbb{P} \otimes \lambda)$. Define $h_0(x) := \mathbb{E}H(x)$ and $h_n(x, x_1, \ldots, x_n) := \frac{1}{n!}\mathbb{E}D^n_{x_1,\ldots,x_n}H(x)$ and assume that H is in $\operatorname{dom}\delta$ (the domain of the operator δ), that is

$$\sum_{n=0}^{\infty}(n+1)! \int \tilde{h}_n^2 \, d\lambda^{n+1} < \infty.$$

Then the Kabanov-Skorohod integral $\delta(H)$ of H is given by

$$\delta(H) := \sum_{n=0}^{\infty} I_{n+1}(h_n).$$

Theorem (Nualart and Vives '90)

Let $F \in \operatorname{dom} D$ *and* $H \in \operatorname{dom} \delta$. *Then*

$$\mathbb{E} \int (D_x F) H(x) \, \lambda(dx) = \mathbb{E} F \delta(H).$$

Theorem (Picard '96; L. and Penrose '11)

Let $H \in L^1_\eta(\mathbb{P} \otimes \lambda) \cap L^2_\eta(\mathbb{P} \otimes \lambda)$ *be in the domain of* δ *and have representative* h. *Then*

$$\delta(H) = \int h(\eta - \delta_x, x) \, \eta(dx) - \int h(\eta, x) \, \lambda(dx) \quad \mathbb{P}\text{-a.s.}$$

Definition

The domain dom L of the Ornstein-Uhlenbeck generator L is given by all $F \in L^2_\eta$ satisfying

$$\sum_{n=1}^{\infty} n^2 n! \|f_n\|_n^2 < \infty.$$

For $F \in \operatorname{dom} L$ one defines

$$LF := -\sum_{n=1}^{\infty} n I_n(f_n).$$

The (pseudo) inverse L^{-1} of L is given by

$$L^{-1}F := -\sum_{n=1}^{\infty} \frac{1}{n} I_n(f_n).$$

3. Normal approximation: General results

Definition

The Wasserstein distance between the laws of two random variables Y_1, Y_2 is defined as

$$d_W(Y_1, Y_2) = \sup_{h \in \mathrm{Lip}(1)} |\mathbb{E}h(Y_1) - \mathbb{E}h(Y_2)|.$$

Theorem (Peccati, Solé, Taqqu and Utzet '10)

Let $F \in \operatorname{dom} D$ such that $\mathbb{E}F = 0$ and N be standard normal. Then

$$d_W(F, N) \le \mathbb{E}\left|1 - \int (D_x F)(-D_x L^{-1} F)\, \lambda(dx)\right| + \mathbb{E} \int (D_x F)^2 |D_x L^{-1} F|\, \lambda(dx).$$

Definition

The Kolmogorov distance between the laws of two random variables Y_1, Y_2 is defined by

$$d_K(Y_1, Y_2) = \sup_{x \in \mathbb{R}} |\mathbb{P}(Y_1 \leq x) - \mathbb{P}(Y_2 \leq x)|.$$

Theorem (Schulte '12)

For $F \in \operatorname{dom} D$ and N standard normal

$$d_K(F,N) \le \left[\mathbb{E}\left(1 - \int (D_x F)(-D_x L^{-1} F)\lambda(dx)\right)^2\right]^{1/2}$$
$$+ 2c(F)\left[\mathbb{E}\int (D_x F)^2 (D_x L^{-1} F)^2 \lambda(dx)\right]^{1/2}$$
$$+ \sup_{t\in\mathbb{R}} \mathbb{E}\int D_x \mathbf{1}\{F > t\}(D_x F)|D_x L^{-1} F|\,\lambda(dx),$$

where

$$c(F) = \left[\mathbb{E}\int (D_x F)^4 \lambda(dx)\right]^{1/2}$$
$$+ \left[\mathbb{E}\int (D_x F)^2 (D_y F)^2 \lambda^2(d(x,y))\right]^{1/4} \left((\mathbb{E}F^4)^{1/4} + 1\right).$$

Theorem (Hug, L. and Schulte '13)

Let $F \in L^2_\eta$ have the chaos expansion

$$F = \mathbb{E}F + \sum_{n=1}^{\infty} I_n(f_n)$$

and assume that $\mathbb{V}\mathrm{ar}\, F > 0$. Assume that there are $a > 0$ and $b \geq 1$ such that

$$\int |(f_m \otimes f_m \otimes f_n \otimes f_n)_\sigma| \, d\lambda^{|\sigma|} \leq \frac{a\, b^{m+n}}{(m!)^2 (n!)^2}$$

for all $\sigma \in \Pi_{mn}$. Let N be a standard Gaussian random variable. Then, under an additional integrability assumption,

$$d_W\left(\frac{F - \mathbb{E}F}{\sqrt{\mathbb{V}\mathrm{ar}\, F}}, N\right) \leq 2^{\frac{15}{2}} \sum_{n=1}^{\infty} n^{17/2} \frac{b^n}{\lfloor n/14 \rfloor !} \frac{\sqrt{a}}{\mathbb{V}\mathrm{ar}\, F}.$$

Theorem (L., Peccati and Schulte '14)

Let $F \in \mathrm{dom}\, D$ *be such that* $\mathbb{E}F = 0$ *and* $\mathbb{V}\mathrm{ar}\, F = 1$, *and let* N *be standard Gaussian. Then,*

$$d_W(F,N) \le \gamma_1 + \gamma_2 + \gamma_3,$$

where

$$\gamma_1^2 := 16 \int \left[\mathbb{E}(D_{x_1}F)^2(D_{x_2}F)^2\right]^{1/2} \left[\mathbb{E}(D^2_{x_1,x_3}F)^2(D^2_{x_2,x_3}F)^2\right]^{1/2} d\lambda^3,$$

$$\gamma_2^2 := \int \mathbb{E}(D^2_{x_1,x_3}F)^2(D^2_{x_2,x_3}F)^2\, d\lambda^3,$$

$$\gamma_3 := \int \mathbb{E}|D_xF|^3\, \lambda(dx).$$

Theorem (L., Peccati and Schulte '14)

For $F, G \in \mathrm{dom}\, D$ *with* $\mathbb{E}F = \mathbb{E}G = 0$*, we have*

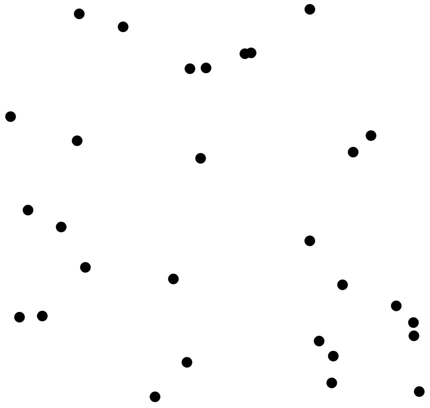
$$\mathbb{E}\left(\mathbb{C}\mathrm{ov}(F,G) - \int (D_x F)(-D_x L^{-1} G)\,\lambda(dx) \right)^2$$

$$\leq 3 \int \left[\mathbb{E}(D^2_{x_1,x_3}F)^2 (D^2_{x_2,x_3}F)^2\right]^{1/2} \left[\mathbb{E}(D_{x_1}G)^2 (D_{x_2}G)^2\right]^{1/2} d\lambda^3$$

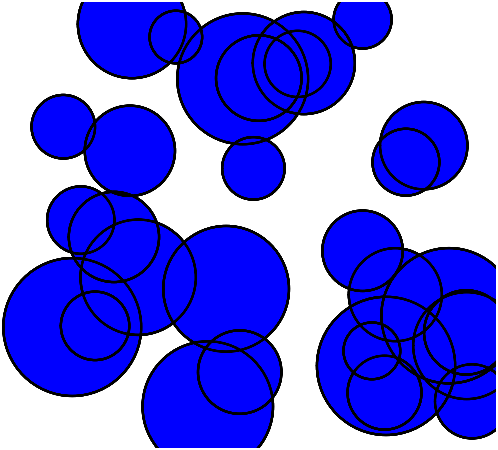
$$+ \int \left[\mathbb{E}(D_{x_1}F)^2 (D_{x_2}F)^2\right]^{1/2} \left[\mathbb{E}(D^2_{x_1,x_3}G)^2 (D^2_{x_2,x_3}G)^2\right]^{1/2} d\lambda^3$$

$$+ \int \left[\mathbb{E}(D^2_{x_1,x_3}F)^2 (D^2_{x_2,x_3}F)^2\right]^{1/2} \left[\mathbb{E}(D^2_{x_1,x_3}G)^2 (D^2_{x_2,x_3}G)^2\right]^{1/2} d\lambda^3.$$

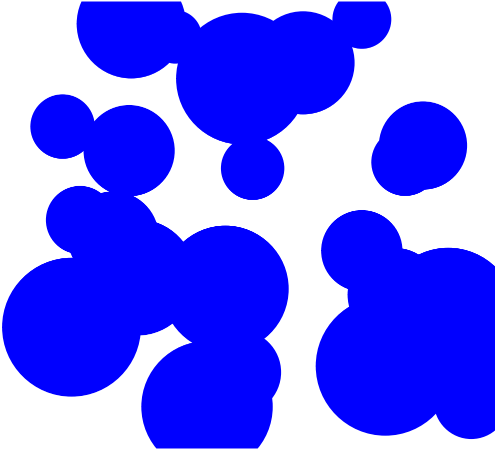
4. The Boolean model



4. The Boolean model



4. The Boolean model



Setting

η is a Poisson process on $\mathcal{K}^d$ (the space of convex bodies) with intensity measure Λ of the translation invariant form

$$\Lambda(\cdot) = \gamma \iint \mathbf{1}\{K + x \in \cdot\}\, dx\, \mathbb{Q}(dK),$$

where $\gamma > 0$, and $\mathbb{Q}$ is a probability measure on $\mathcal{K}^d$ with $\mathbb{Q}(\{\emptyset\}) = 0$ and

$$\int V_d(K + C)\mathbb{Q}(dK) < \infty, \quad C \subset \mathbb{R}^d \text{ compact}.$$

Definition

The Boolean model (based on η) is the random closed set

$$Z := \bigcup_{K \in \eta} K.$$

Remark

The Boolean model is stationary, that is

$$Z + x \stackrel{d}{=} Z, \quad x \in \mathbb{R}^d.$$

Remark

The intersection $Z \cap W$ of the Boolean model Z with a convex set $W \in \mathcal{K}^d$ belongs to the convex ring $\mathcal{R}^d$, that is, $Z \cap W$ is a finite union of convex bodies.

4.1 Mean values

Theorem

Let $\psi : \mathcal{R}^d \to \mathbb{R}$ be measurable, additive, translation invariant and locally bounded. Let $W \in \mathcal{K}^d$ with $V_d(W) > 0$. Then the limit

$$\delta_\psi := \lim_{r\to\infty} \frac{\mathbb{E}\psi(Z \cap rW)}{V_d(rW)}$$

exists and is given by

$$\delta_\psi = \mathbb{E}[\psi(Z \cap [0,1]^d) - \psi(Z \cap \partial^+[0,1]^d)],$$

where $\partial^+[0,1]^d$ is the upper right boundary of the unit cube $[0,1]^d$. Moreover, due to ergodicity, there is a pathwise version of this convergence.

Example

The intrinsic volumes $V_0, \dots, V_d$ have all properties required by the theorem. For $K \in \mathcal{K}^d$, they are defined by the Steiner formula

$$V_d(K + rB^d) = \sum_{j=0}^{d} r^j \kappa_j V_{d-j}(K), \quad r \geq 0,$$

where κ_j is the volume of the Euclidean unit ball in $\mathbb{R}^j$. The intrinsic volumes can be additively extended to the convex ring.

Definition

Let Z_0 be typical grain, that is, a random closed set with distribution $\mathbb{Q}$ and define

$$v_i := \mathbb{E}\, V_i(Z_0) = \int V_i(K)\mathbb{Q}(dK), \quad i = 0, \dots, d,$$

Example

The volume fraction of Z is defined by

$$p := \mathbb{E} V_d(Z \cap [0,1]^d) = \mathbb{P}(0 \in Z)$$

and given by the formula

$$p = 1 - \exp[-\gamma v_d].$$

Note that

$$\mathbb{E}\lambda_d(Z \cap B) = p\lambda_d(B), \quad B \in \mathcal{B}(\mathbb{R}^d),$$

that is $\delta_{V_d} = p$.

Example

For any $W \in \mathcal{K}^d$,

$$\mathbb{E} V_{d-1}(Z \cap W) = V_d(W)(1-p)\gamma v_{d-1} + V_{d-1}(W)p.$$

Therefore the surface density $\delta_{V_{d-1}}$ of Z is given by the formula

$$\delta_{V_{d-1}} = (1-p)\gamma v_{d-1}.$$

Definition

The Boolean model is called isotropic if $\mathbb{Q}$ is invariant under rotations or, equivalently, if

$$Z \overset{d}{=} \rho Z$$

for all rotations $\rho : \mathbb{R}^d \to \mathbb{R}^d$.

Theorem (Miles '76, Davy '78)

Assume that Z is isotropic and let $j \in \{0, \ldots, d\}$. Then

$$\mathbb{E} V_j(Z \cap W) - V_j(W) = -(1-p) \sum_{k=j}^{d} V_k(W) P_{j,k}(\gamma v_j, \ldots, \gamma v_{d-1})$$

for any $W \in \mathcal{K}^d$, where the polynomials $P_{j,k}$ are defined below. In particular

$$\delta_{V_j} = -(1-p) P_{j,d}(\gamma v_j, \ldots, \gamma v_{d-1}).$$

Remark

The first formula can be extended to more general additive functionals.

Definition

For $j \in \{0, \dots, d-1\}$ and $k \in \{j, \dots, d\}$ define a polynomial $P_{j,k}$ on $\mathbb{R}^{d-j}$ of degree $k - j$ by

$$P_{j,k}(t_j, \dots, t_{d-1}) := \mathbf{1}\{k = j\} + c_j^k \sum_{s=1}^{k-j} \frac{(-1)^s}{s!} \sum_{\substack{m_1,\dots,m_s=j \\ m_1+\dots+m_s=sd+j-k}}^{d-1} \prod_{i=1}^{s} c_d^{m_i} t_{m_i},$$

where

$$c_j^k := \frac{k!\kappa_k}{j!\kappa_j}.$$

4.2 Covariance structure

Definition

For $p \geq 1$ the integrability assumption $IA(p)$ holds if

$$\mathbb{E} V_i(Z_0)^p < \infty, \quad i = 0, \dots, d,$$

Assumption

$IA(2)$ is assumed to hold throughout the rest of this section.

Definition

Let

$$C_W(x) := V_d(W \cap (W + x)), \quad x \in \mathbb{R}^d,$$

be the covariogram of $W \in \mathcal{K}^d$ and

$$C_d(x) := \mathbb{E} V_d(Z_0 \cap (Z_0 + x)), \quad x \in \mathbb{R}^d,$$

the mean covariogram of the typical grain.

Theorem

We have

$$\mathbb{P}(0 \in Z, x \in Z) - p^2 = (1-p)^2 \left(e^{\gamma C_d(x)} - 1\right), \quad x \in \mathbb{R}^d,$$

$$\mathbb{V}\mathrm{ar}(V_d(Z \cap W)) = (1-p)^2 \int C_W(x) \left(e^{\gamma C_d(x)} - 1\right) dx, \; W \in \mathcal{K}^d.$$

Definition

Let $W \in \mathcal{K}^d$ satisfy $V_d(W) > 0$. The asymptotic covariances of the intrinsic volumes are defined by

$$\sigma_{i,j} := \lim_{r\to\infty} \frac{\mathbb{C}\mathrm{ov}(V_i(Z \cap rW), V_j(Z \cap rW))}{V_d(rW)}, \quad i,j = 0,\dots,d.$$

Example

As it is well-known that

$$\lim_{r(W)\to\infty} \frac{C_W(x)}{V_d(W)} = 1,$$

where $r(W)$ denotes the inradius of W, it follows that

$$\sigma_{d,d} = (1-p)^2 \int \big(e^{\gamma C_d(x)} - 1\big)\,dx.$$

Theorem (Hug, L. and Schulte '13)

The asymptotic covariances $\sigma_{i,j}$ exist. Moreover, there is a constant $c > 0$ depending only on the dimension and Λ, such that

$$\left| \frac{\mathbb{C}\mathrm{ov}(V_i(Z \cap rW), V_j(Z \cap rW))}{V_d(rW)} - \sigma_{i,j} \right| \leq \frac{c}{r(W)}.$$

Moreover, this rate of convergence is optimal.

Main ideas of the Proof:

The Fock space representation of Poisson functionals (cf. L. and Penrose '11) gives for any $F, G \in L^2_\eta$, that

$$\mathbb{C}\mathrm{ov}(F,G) = \sum_{n=1}^{\infty} \frac{1}{n!} \int (\mathbb{E} D^n_{K_1,\dots,K_n} F)(\mathbb{E} D^n_{K_1,\dots,K_n} G)\, \Lambda^n(d(K_1,\dots,K_n)).$$

For an additive functional $\psi : \mathcal{R}^d \to \mathbb{R}$ define

$$f_{\psi,W}(\mu) := \psi(Z(\mu) \cap W),$$

where $Z(\mu)$ is the union of all grains charged by the counting measure μ. Then

$$D^n_{K_1,\dots,K_n} f_{\psi,W}(\mu) = (-1)^n(\psi(Z(\mu) \cap K_1 \cap \ldots \cap K_n \cap W) - \psi(K_1 \cap \ldots \cap K_n \cap W)).$$

One can prove that for all $n \in \mathbb{N}$ and $K_1, \dots, K_n \in \mathcal{K}^d$

$$|\mathbb{E} D^n_{K_1,\dots,K_n} f_{\psi,W}(\eta)| \le \beta(\psi) \sum_{i=0}^{d} V_i(K_1 \cap \dots \cap K_n \cap W),$$

where the constant $\beta(\psi)$ does only depend on ψ, Λ and the dimension. Applying the (new) integral geometric inequalities

$$\int V_k(A \cap (K + x))dx \le \beta_1 \sum_{i=0}^{d} V_i(A) \sum_{r=k}^{d} V_r(K), \quad A \in \mathcal{K}^d,$$

where β_1 depends only on the dimension, and

$$\int \sum_{k=0}^{d} V_k(A \cap K_1 \cap \dots \cap K_n) \Lambda^n(d(K_1, \dots, K_n)) \le \alpha^n \sum_{k=0}^{d} V_k(A),$$

where $\alpha := \gamma(d+1)\beta_1 \sum_{i=0}^{d} \mathbb{E} V_i(Z_0)$ allows to use dominated convergence to derive the result.

Theorem (Hug, L. and Schulte '13)

Assume that $IA(4)$ holds. Let $W \in \mathcal{K}^d$ with $r(W) \geq 1$, $i \in \{0, \ldots, d\}$, and N be a centred Gaussian random variable. Then, for all $W \in \mathcal{K}^d$ with sufficiently large inradius,

$$d_W\big(\mathbb{V}\mathrm{ar}(V_i(Z \cap W))^{-1}(V_i(Z \cap W) - \mathbb{E}V_i(Z \cap W)), N\big) \\ \leq c V_d(W)^{-1/2},$$

where the constant $c > 0$ depends only on Λ and the dimension. If only $IA(2)$ holds, then we still have convergence in distribution.

Main ideas of the Proof:

Let

$$f_n(K_1,\dots,K_n) := \frac{(-1)^n}{n!}\big(\mathbb{E}V_i(Z\cap K_1\cap\dots\cap K_n\cap W) - V_i(K_1\cap\dots\cap K_n\cap W)\big)$$

be the kernels of the chaos expansion of $V_i(Z\cap W)$. Bound

$$\int |(f_m\otimes f_m\otimes f_n\otimes f_n)_\sigma|\,d\Lambda^{|\sigma|}$$

using the previous integral geometric inequalities and apply one of the previous theorems.

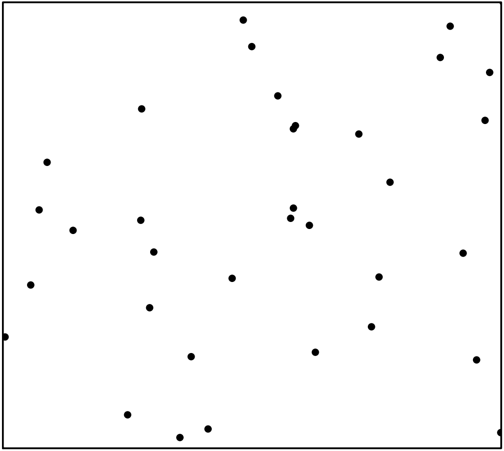
Remark

The above result remains true for any additive, locally bounded and measurable (but not necessarily translation invariant) functional ψ, provided that the variance of $\psi(Z \cap W)/V_d(W)^{-1/2}$ does not degenerate for large W.

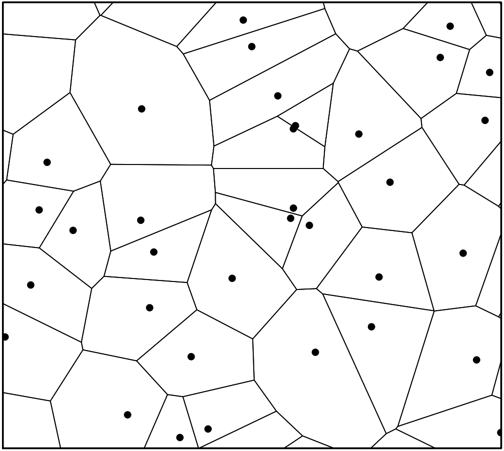
Theorem (Hug, L. and Schulte '13)

If the typical grain Z_0 has nonempty interior with positive probability, then the covariance matrix $(\sigma_{i,j})$ is positive definite.

5. Poisson-Voronoi tessellation



5. Poisson-Voronoi tessellation



Setting

η is a stationary Poisson process on $\mathbb{R}^d$ with unit intensity.

Definition

The Poisson-Voronoi tessellation is the collection of all cells

$$C(x, \eta) = \{y \in \mathbb{R}^d : \|x - y\| \leq \|z - y\|, z \in \eta\}, \quad x \in \eta.$$

For $k \in \{0, \ldots, d\}$ let X^k denote the system of all k-faces of the tessellation.

Theorem (Avram and Bertsimas '93, Penrose and Yukich'05, L., Peccati and Schulte '14)

Fix $W \in \mathcal{K}^d$ *and let*

$$V_r^{(k,i)} := \sum_{G \in X} V_i(G \cap rW), \quad r > 0,$$

where $k \in \{0, \ldots, d\}$, $i \in \{0, \ldots, \min\{k, d-1\}\}$. *There are constants* $c_{k,i}$, *such that*

$$d_W\left(\frac{V_r^{(k,i)} - \mathbb{E}\, V_r^{(k,i)}}{\sqrt{\mathbb{V}\mathrm{ar}\, V_r^{(k,i)}}}, N\right) \leq c_{k,i} r^{-d/2}, \quad r \geq 1.$$

A similar result holds for the Kolmogorov distance.

Main ideas of the Proof: Use the stabilizing properties of the Poisson Voronoi tessellation to show that

$$\sup_{r\geq 1}\sup_{x}\mathbb{E}\Big|D_x V_r^{(k,i)}\Big|^5+\sup_{r\geq 1}\sup_{x,y}\mathbb{E}\Big|D^2_{x,y} V_r^{(k,i)}\Big|^5<\infty$$

and, for $q:=1/20$,

$$\int_{rW}\mathbb{P}(D_x V_r^{(k,i)}\neq 0)^q dx\leq cr^d,$$

$$\int_{rW}\left(\int_{rW}\mathbb{P}(D^2_{x,y} V_r^{(k,i)}\neq 0)^q dx\right)^2 dy\leq cr^d.$$

Show with other methods that

$$\liminf_{r\to\infty} r^{-d}\,\mathbb{V}\mathrm{ar}\, V_r^{(k,i)} > 0.$$

Combine this with (a consequence of) the second order Poincaré inequality to conclude the result.

6. References

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