V. The Curved Cartan Complex

- Given an infinitesimally trivialized G-action on a category $\mathcal{C}$, the C^3 is a specific, Koszul dual model for the (derived) quotient cat. $\mathcal{C}_{G/\hat{G}}$.

- The "obvious" model is the crossed product $(G/\hat{G}) \ltimes \mathcal{C}$. The group algebra of $G/\hat{G}$ is the chain complex with convolution product and enhances the Hom spaces of $\mathcal{C}$.

- The same "obvious" model can be interpreted as $H_*(BG; \underline{\mathcal{C}})$, but with the stack BG (needs care)

- The Koszul dual C^3 replaces $H_* G$ in the Homs by $H^*(BG)$. $\mathbb{Z}/2$ collapse globalizes the Patter, allowing for non-perturbative information.

- This recovers the correct gauge theory of a point or vector space (A-model with topological twist)

(In particular, the Casimir twist gives a theory isomorphic to the B-model.)

Definition

Recall that if G acts on an algebra A, then $(G \ltimes A)\text{-mod} = (A\text{-mod})^G$. For simplicity, focus on this

- G acts differentiably on a (dg) algebra A
- $\mathcal{L} : \mathfrak{g} \longrightarrow HZ^1(A)$ is the Lie algebra action

  $\uparrow$ Hochschild 1-cocycles (derivations)
- $\iota : \mathfrak{g} \longrightarrow HCH^0(A)$ (0-cochains), the trivialization of the action,

such that:

$$(\iota, \mathcal{L}) : \big(\underset{(-1)}{\mathfrak{g}} \xrightarrow{=} \underset{(0)}{\mathfrak{g}}\big) \longrightarrow \big(\ldots HCH^{-1} \to HCH^0 \to HZ^1\big)$$

is a morphism of dgla's, equivariant for ~~Ha~~ G.

From this we build the following curved dga:

$$G \ltimes (A \otimes \mathrm{Sym}\,\mathfrak{g}^*),\ d = d_A + \xi^a \cdot \iota(\xi_a),\ W = \xi_a(\delta_1) \otimes \xi^a$$

(caution!)

satisfying the curved dga relation $d^2 = [W, \cdot]$.

Notes: ξ_a basis of $\mathfrak{g}$, ξ^a dual basis, $\xi_a(\delta_1) \in C^{-\infty}(G)$.

Caution: The placement of $\iota(\xi)$ in the differential is predicated on its Hochschild degree being $= 1$, as in the standard Cartan example. In general, $\iota(\xi)$ gets distributed according to Hochschild degree leading to a (curved) A_∞ algebra

Examples I

Cartan Example: G acts on smooth X, $A = (\Omega^\bullet(X), d)$

$\mathcal{L}_\xi$ = Lie derivative, $\iota(\xi)$ = contraction, Hoch. deg. $\iota = 1$;

$$d_{\text{Cartan}}^2 = \xi^a \cdot \mathcal{L}(\xi_a) = [\underset{\hat{\mathfrak{g}}^*}{\xi^a} \otimes \xi_a(\delta_1), \cdot].$$

Remark: To compute $H^*_G(X)$ we would take G-invariants.

Trivial Example: $A = \mathbb{C}$. Get $G \ltimes \operatorname{Sym} \mathfrak{g}^*$, $W = \xi^a \otimes \xi_a(\delta_1)$

Modules over it are equivalent to $(\operatorname{Sym} \mathfrak{g}^*)^G = H^*(BG)$-mods

For instance, if $G = T$,

$$\operatorname{Spec}(T \ltimes \operatorname{Sym} \mathfrak{t}^*) = \coprod_{\lambda \in \operatorname{char}(T)} \mathfrak{t}_\lambda, \quad W_\lambda = \lambda \quad \xi \mapsto \lambda(\xi)$$

So curved modules "live" on the zero-sheet.

This example leads to string topology of BG, a not-quite-finite 2dim. TQFT.

Non-comm. example: $A = U(\mathfrak{g})$, G acts by ad.

So $\mathcal{L}(\xi) = [\xi, \cdot] : U \to U$, $\iota(\xi) = \xi \in U$.

Get $G \ltimes (U(\mathfrak{g}) \otimes \operatorname{Sym} \mathfrak{g}^*)$, $W = (\xi_a(\delta_1) - \xi_a) \otimes \xi^a$.
$\hookrightarrow$ deg 2

Modules over this are equivalent to G-representations;
the curvature forces a cancellation.

(Best seen in the Koszul dual model

$$G \ltimes (\Lambda^\bullet \mathfrak{g} \otimes A), \quad d_A + \partial,$$

Lie algebra diff

ι defines algebra action or differential $\mathfrak{g} \to A$

Examples II: the torus

The regular Representation of T/T_{loc}:

This is $Coh(T^\vee_{\mathbb{C}})$ as a module category over itself.

The action of T on the algebra $\mathbb{C}[T^\vee_{\mathbb{C}}]$ is concealed in the Poincaré bundle $\mathcal{P} \to T \times T^\vee_{\mathbb{C}}$, which is flat and multiplicative along T: $\mathcal{P}_t \otimes \mathcal{P}_s \xrightarrow{\sim} \mathcal{P}_{ts}$.
$\mathcal{P}_t = \mathcal{P}|_{t \times T^\vee}$

[The action of $t \in T$ is $\otimes \mathcal{P}_t$; the trivialization comes from following the connection on a path $1 \rightsquigarrow t$]

Fact: The crossed product algebra $T \ltimes \mathbb{C}[T^\vee_{\mathbb{C}}]$ is the algebra of functions on $\mathfrak{t}^\vee_{\mathbb{C}}$. The C^3 is $\mathbb{C}[\mathfrak{t} \times \mathfrak{t}^\vee]$ with potential $W = \xi^a z_a$. Knörrer periodicity identifies the curved module category $\cong$ Vect.

Landau-Ginzburg models $Y \to T^\vee_{\mathbb{C}}$, $W: Y \to \mathbb{C}$.

The crossed product $T \ltimes Y = \tilde{Y}$, the covering pulled back from $\mathfrak{t}^\vee_{\mathbb{C}} \to T^\vee_{\mathbb{C}}$. The CCC is $\tilde{Y} \times \mathfrak{t}$ with potential $W + \xi^a z_a$.

This is equivalent to $(Y_1, W|_{Y_1})$ where $Y_1 \subset Y$ is the (scheme-theoretic) fiber.

Casimir Curvings vs Topological Twists

Gromov-Witten theory of X has a natural family of deformations, "topological twists", over $H^*(X)$.

Geometrically: $\overline{X}_g^n$:= space of stable ~~maps~~ curves

n = #points, g = genus

$ev: \overline{X}_g^{n+1} \to X$ last eval, $\alpha \in H^*(X)$

Define the α-twisted invariant by using

$$\exp\left(\int_{\overline{X}_g^{n+1}}^{\overline{X}_g^n} ev^*\alpha\right) \cap \left[\overline{X}_g^n\right]_{\text{virtual}}$$

in place of the virtual fundamental class in GW.

In the gauge theory of a point (A-model), these twists lead to Witten's integrals on the moduli of G bundles.

They come from classes in BG.

For instance, quad. Casimir (in H^4) $\to$ symplectic volume.

Q: How do you twist the category of boundary states?

The CCC (of anything) allows for additional curvings, from $(\mathrm{Sym}\,\mathfrak{g}^*)^G \cong H^*(BG)$; these are central elements.

Q: What effect do these have on the fixed point cat.?

The two questions answer each other.

The quadratic Casimir

has a dramatic effect on the quotient category $\mathrm{Vect}_{G/\hat{G}}$: it renders it semisimple.

Theorem The $\frac{1}{2}\xi^2$-curved CC is semisimple with one generator for each integral regular co-adjoint orbit in $\mathfrak{g} \cong \mathfrak{g}^*$. A generator for each orbit is the respective Atiyah-Bott-Shapiro Thom class.

Remark The category is thus supported on the regular part of $\mathfrak{g}$, where the CCC is Morita equivalent to the Weyl quotient of its max torus version.

Remark. For $\sum_1^n x_i^2$ on $\mathbb{C}^n$, the MF category is equivalent to modules over the Clifford algebra. This Koszul equivalence is mediated by the curved complex

$$\mathrm{Cliff}^{ev} \underset{\psi(x)\cdot}{\overset{\psi(x)\cdot}{\rightleftarrows}} \mathrm{Cliff}^{odd}$$

Same for a Morse-Bott function.

Theorem. With curvature $\Phi = \frac{1}{2}\xi^2 + \varepsilon P$ $\left(P \in (\mathrm{Sym}^{\geq 2}\mathfrak{g}^*)^G\right)$ ~~the B model theory with~~ and trace induced from the volume form on $\mathfrak{g}$, the CCC generates Witten's topological Yang-Mills with top. twisting $\frac{1}{2}$(quad. cas) $+ \varepsilon P$.

Illustration for a torus

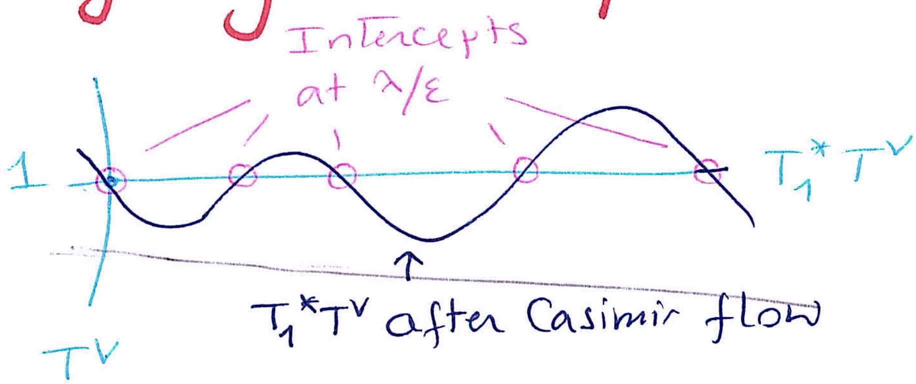
Recall $\mathrm{Spec}(CCC) = \coprod \mathfrak{t}_\lambda$,

$$W(z) = \lambda(z) + \tfrac{1}{2} z^2$$

$\Rightarrow$ Morse critical point $z = -\lambda$ on sheet $\mathfrak{t}_\lambda$

Remark: The critical points "come from infinity":
With $\frac{\varepsilon}{2} z^2$, we get $z = -\lambda/\varepsilon$. "Non-perturbative"!

Lagrangian interpretation



The fixed-point category is Hom from $T_1^* T^\vee$ in Kapustin-Rozansky theory.

~~Deformation of the trivial rep Vect_1~~

~~Deformations of Vect_1 as a $(\mathrm{Coh}(T^\vee), \otimes)$-module are controlled by End(Id) in $\mathrm{Ext}_{(\mathrm{Coh}(T^\vee),\otimes)}(V_1, V_1)$.~~

~~The latter is $(\Lambda T_1^* T^\vee)$-modules with $\otimes_{\mathbb{C}}$, so the unit is $\mathbb{C}$ and $= \mathrm{Sym}\, T_1 T^\vee \cong \mathrm{Sym}\, \mathfrak{t}^*$.~~

~~For general G we get similarly $(\mathrm{Sym}\, \mathfrak{g}^*)^G$.~~

Deformation of the trivial rep. Vect_1

As a $(\mathrm{Coh}(T^\vee), \otimes)$-module, $\mathbb{Z}/2$ graded defs. of the 'fiber functor' Vect_1 are controlled by $\mathrm{REnd}(\mathbb{1})$, the tensor unit in $\mathrm{Ext}_{(\mathrm{Coh}\,T^\vee, \otimes)}(\mathrm{Vect}_1, \mathrm{Vect}_1)$

The latter category is $\Lambda^\bullet(\mathfrak{t}_1^* T^\vee)$-modules with (Hopf) tensor structure over $\mathbb{C}$.

The tensor unit is $\mathbb{C}$, so $\Rightarrow = \mathrm{Sym}(\mathfrak{t}_1 T^\vee) = \mathrm{Sym}\,\mathfrak{t}^*$.

For a general semi-simple group one can show the answer is $(\mathrm{Sym}\,\mathfrak{g}^*)^G$.

[The degree 2 part gives $\mathbb{Z}$-graded deformations.]

Re-interpretation of Extra-curved Cartan Cx

$$G \ltimes (A \otimes \mathrm{Sym}\,\mathfrak{g}^*), \quad d_{cc}, \quad W = \xi^a \otimes \xi_a(\delta_1) + P(\xi)$$

computes the (derived, non-perturbative) (twisted) co-invariant category $A \otimes \mathrm{Vect}_P$.

Example For $\tau \in T^\vee, \tau \neq 1$, $\mathrm{Vect}_1 \underset{T/T}{\otimes} \mathrm{Vect}_\tau = 0$,

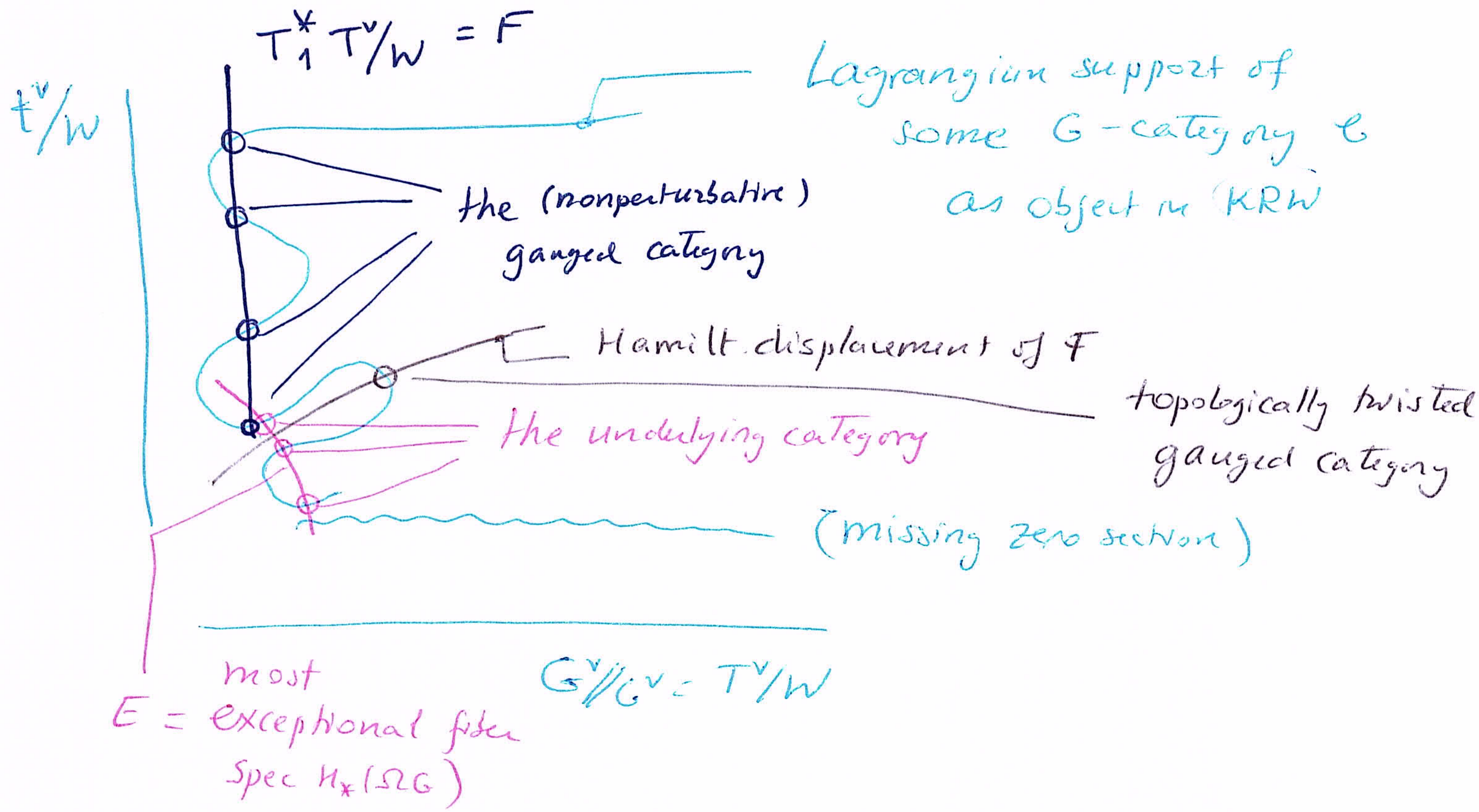
the representations are disjoint. But after Casimir twist,

$$\mathrm{Vect}_1 \underset{T/T}{\otimes} \underset{\text{Casimir}}{\mathrm{Vect}_{\tau, (P)}} \cong \bigoplus \mathrm{Vect} \quad \text{because} \quad \lambda(\xi) + (\log \tau)(\xi) + \tfrac{1}{2}\xi^2$$

has a Morse critical point on each $\mathfrak{t}_\lambda$.

Interpretation of the BFM $T^*(G^\vee/G^\vee)$

Recall the affine blow-up of $T^*T^\vee/W$, also $(T^*G^\vee)^{reg}//G^\vee$ and $\operatorname{Spec} H_*^G(\Omega G)$ (Theorem)



We can explore the entire BFM space via Hamiltonian displacements of the fiber F; this corresponds to computing (all) topologically twisted gaugings of $\mathcal{C}$.

In torus case: linear Hamiltonians suffice and we get the spectral decomposition of $\mathcal{C}$ over $T^\vee$.

If $\operatorname{Supp} \mathcal{C}$ is not closed or has essential singularities, then some topologically twisted gaugings of $\mathcal{C}$ will be ill-behaved.

A lower-dimensional analogue

This is equivariant homology/cohomology of a chain complex with infinitesimally trivialized G action.

Thus $H^*_G(V^\cdot)$ is a module over $H^*(BG) = (\mathrm{Sym}\,\mathfrak{g}^*)^G$ so fibers naturally over $\mathfrak{g}/G = \mathfrak{t}/W$.

The derived fiber over 0, $C^*(BG; V^\cdot) \overset{L}{\underset{C^*BG}{\otimes}} \mathbb{C}$ is quasi-isomorphic to $V^\cdot$, the original complex.

We can put on the Kapustin goggles and double the base to $H^G_*(G)$ to get an odd version of BFM space.

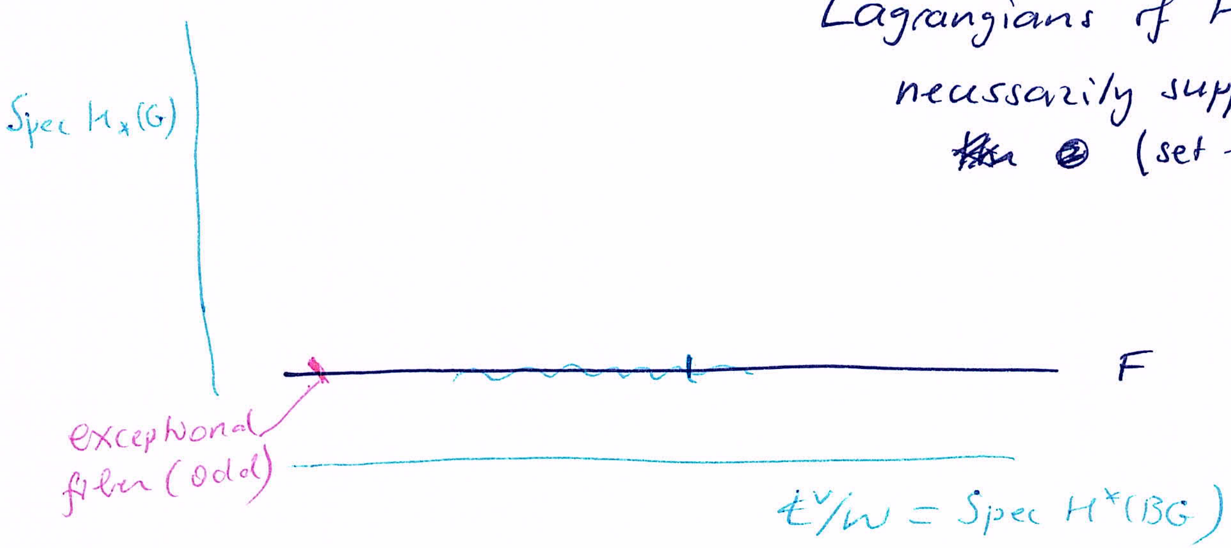
This is an affine blow-up of $(\mathrm{Sym}\,\mathfrak{t}^* \otimes \Lambda^\cdot \mathfrak{t})^W$

[adjoin $\alpha^{\text{odd}\ \cancel{\text{even}}} / \alpha^{\text{even}\ \cancel{\text{odd}}}$]

'string topology' of BG

$HH^*(H^*(BG))$

Lagrangians of $H^*_G(V^\cdot)$ are necessarily supported on F (set-theoretically)



Further directions

There is a Loop Group analogue of the story the Curved Cartan complex and BFM space, which is roughly the blow-up of $(T \times T^\vee)/W$.

It should be a ~~2d~~ dimensional reduction along the circle of 3D A-style gauge theory; it is definitely connected to K-theoretic GW invariants. Indeed, for a point one computes the "Verlinde" version of Witten's integration formulas.

The collection of all these theories should represent (S^1-reduced) 4-dimensional pure gauge theory for G. Langlands duality ought to be a salient feature here.

Finally one can guess that k-dimensional, A-style gauge theories are "described" by k-categorifications of $H_*^G(\Omega^k BG) = H_*^G(\Omega^{k-1}G)$.

$H_*^G(G)$ if $k=1$

This gives at least a starting point in high dimension.