V. The Curved Cartan Complex

• Given an infinitesimally trivialized $G$-action on a category $\mathcal{C}$, the $C^3$ is a specific, Koszul dual model for the (derived) quotient cat. $\mathcal{C}/G$.

• The "obvious" model is the crossed product $(G/\hat{G}) \times \mathcal{C}$. The group algebra of $G/\hat{G}$ is the chain complex with convolution product and enhances the Hom spaces of $\mathcal{C}$.

• The same "obvious" model can be interpreted as $H_*^* (B\mathcal{G}; \mathcal{C})$, but with the stack $B\mathcal{G}$ (needs care).

• The Koszul dual $C^3$ replaces $H^* G$ in the Hom by $H^* (B\mathcal{G})$. $\mathbb{Z}/2$ collapse globalizes the pattern, allowing for non-perturbative information.

• This recovers the correct gauge theory of a point on vector space (A-model with topological twist) (In particular, the Casimir twist gives a theory isomorphic to the B-model.)
Definition

Recall that if $G$ acts on an algebra $A$, then $(G \times A) \text{-mod} = (A \text{-mod})^G$. For simplicity, focus on this:

- $G$ acts differentiably on a (dg) algebra $A$.
- $L : g \to \mathcal{H}^1(A)$ is the Lie algebra action
- $\mathcal{H}^1(A)$ is Hochschild 1-cohoms (derivations).
- $\mathcal{H}^0(A)$ is Hochschild 0-cohoms, the trivialization of the action.

Such that:

$$(L, \mathcal{H}) : \left( \begin{array}{c} g \\ \mathcal{H}^1 \\ \mathcal{H}^0 \end{array} \right) \to \left( \begin{array}{c} \mathcal{H}
\end{array} \right)$$

is a morphism of dg $\mathcal{A}$'s, equivariant for $G$.

From this we build the following curved dga:

$$G \times (A \otimes \text{Sym} g^*)$$

$d = d_A + \xi^a \cdot \mathcal{L}_{\delta^a}$, $\mathcal{W} = \xi^a(\delta^a) \otimes \xi^a$ (caution!)

Satisfying the curved dga relation $d^2 = [\mathcal{W}, \cdot]$. 

Notes: $\xi^a$ basis of $g$, $\xi^a$ dual basis, $\xi^a(\delta^a) \in C^\infty(G)$.

Caution: The placement of $\mathcal{W}(\xi)$ in the differential is predicated on its Hochschild degree being $1$, as in the standard Cartan example. In general, $\mathcal{W}(\xi)$ gets distributed according to Hochschild degree, leading to a (curved) $A\infty$ algebra.
Examples I

Cantian Example: $G$ acts on smooth $X$, $A = (\Omega^*(X), \partial)$
$L_a = \text{Lie derivative}, \ L(x) = \text{contraction}, \ \text{Hoch. deg.} \ x = 1$
$\partial_{\text{Cantian}}^2 = \partial^a \cdot L(x_a) = \left[ \frac{\partial^a \otimes \theta_2(x_i)}{x_i} \right].$

Remark: To compute $H^*_G(X)$ we would take $G$-invariants.

Trivial Example: $A = C$. Get $G \times \text{Sym} \ g^*$, $W = \theta_2 \otimes \theta_2(x_i)$
Modules over it are equivalent to $(\text{Sym} \ g^*)_G = H^*(BG)\text{-mod}$
For instance, if $G = T$,

$$\text{Spec} \ (T \times \text{Sym}^* g) = \coprod \theta_{x_i} \quad \text{where} \quad W_\lambda \xrightarrow{x_i \mapsto \lambda(x_i)} \theta_{x_i}$$

so curved modules "live" on the zero-sheet.

This example leads to Shing topology of $BG$, a not-quite-finite 2dim TQFT.

Non-comm. Example: $A = U(g)$, $G$ acts by Ad.
So $L(x) = [x, .] : U \to U$, $\lambda(x) = x, \ x \in U$.

Get $G \times (U(g) \otimes \text{Sym} \ g^*)$, $W = (\theta_2(x_i) - \theta_2) \otimes \theta_2^a$.

Modules over this are equivalent to $G$-representations;
the curvature forces a cancellation.

(Best seen in the Koszul dual model

$$G \times (\Lambda^*g \otimes A), \ \partial_A + \partial, \ n \text{ defines algebra action}$$

$\text{Lie algebra diff}$ or differential $g \to A$
Examples II: the torus

The regular Representation: $\mathcal{C}^{\infty}$

This is $\text{Coh}(\mathcal{T}_e)$ as a module over itself.

The action of $T$ on the algebra $C[\mathcal{T}_e]$ is concealed in the Poincaré bundle $P \to T \times \mathcal{T}_e$, which is flat and multiplicative along $T$: $P_e \otimes P_s \cong P_{ts}$.

$= P|_{t \times t^v}$

[The action of $t \in T$ is $\otimes P_t$; the trivialization comes from following the connection on a path $1 \to t$.]

Fact: The crossed product algebra $T \times C[\mathcal{T}_e]$ is the algebra of functions on $\mathcal{T}_e$. The $C^3$ is $C[\mathcal{T}_v]$ with potential $W = z^a z_i^a$. Knörrer periodically identifies the curved module category $\cong \text{Vect}$.

Landau–Ginzburg models $Y \to T_e^v$, $W: Y \to C$.

The crossed product $T \times Y = \tilde{Y}$, the covering pulled back from $\mathcal{T}_e^v \to T_e^v$. The $C^3$ is $\tilde{Y} \times t$ with potential $W + z^a z_i^a$.

This is equivalent to $(Y_\lambda, W|_{Y_\lambda})$ where $Y_\lambda$ is the (scheme-theoretic) fiber.
Casimir Curvings vs Topological Twists

Gromov-Witten Theory of $X$ has a natural family of deformations, "topological twists," over $H^*(X)$.

Geometrically: $\mathcal{X}_g^h = \text{space of stable maps}^\text{curves}$

$ev: \mathcal{X}_g^{n+1} \to X$ last eval, $d \in H^*(X)$

Define the $\alpha$-twisted invariant by using

$$
\exp \left( \int_{\mathcal{X}_g^{n+1}} ev^\ast \alpha \right) \cap [\mathcal{X}_g^n]_{\text{virtual}}
$$

in place of the virtual fundamental class in $GW$.

In the gauge theory of a point (A-model), these twist lead to Witten's integrals on the moduli of $G$ bundles.

They come from classes in $BG$.

For instance, quad. Casimir (in $H^4$) $\to$ symplectic volume.

Q: How do you twist the category of boundary states?

The CCC (of anything) allows for additional curvings, from $(\text{Sym}g^x)^G \subseteq H^*(BG)$; these are central elements.

Q: What effect do these have on the fixed point cat.?

The two questions answer each other.
The quadratic Casimir has a dramatic effect on the quotient category \( \text{Vect}_{G/C} \): it renders it semisimple.

**Theorem** The \( \frac{1}{2} z^2 \)-curved CCC is semisimple regular with one generator for each integral co-adjoint orbit in \( g \cong g^* \). A generator for each orbit is the respective Atiyah-Bott-Shapiro Thom class.

**Remark** The category is thus supported on the regular part of \( G \), when the CCC is Morita equivalent to the Weyl quotient of its max torus version.

**Remark.** For \( \sum_{i=1}^{n} x_i^2 \) on \( \mathbb{C}^n \), the MF category is equivalent to modules over the Clifford algebra. This Koszul equivalence is mediated by the curved complex

\[
\text{Cliff}^{\text{ev}} \xrightarrow{\Psi(x)^*} \text{Cliff}^{\text{odd}}
\]

Same for a Morse-Bott function.

**Theorem.** With curvature \( \tilde{Q} = \frac{1}{2} z^2 + EP \) \((\text{Pe}(\text{Sym}^2 g^*) G)\) and trace induced from the Batalin-Vilkovisky volume form on \( G \), the CCC generates Witten's topological Yang-Mills with top. twisting \( \frac{1}{2} (\text{quadratic}) + EP \).
Illustration for a torus

Recall \( \text{Spec} (\mathbb{C}^2) = \bigsimeq \mathbb{C} \),

\[ W(\varepsilon) = \varepsilon^2 + \frac{1}{2} \varepsilon^2 \]

\( \Rightarrow \) Morse critical point \( \varepsilon = -\lambda \) on sheet \( \tau \)

Remark: The critical points "come from infinity":

With \( \varepsilon = \frac{1}{2} \varepsilon^2 \), we get \( \varepsilon = -\lambda / \varepsilon \). "Non-perturbative".

Lagrangian interpretation

Intercepts at \( \lambda / \varepsilon \)

The fixed-point category is Hom from \( T^*_{1^*} T^* \) in Kapustin-Rozansky theory.

Deformation of the trivial rep Vect,

Deformations of \( \text{Vect} \) as a \((\text{Coh}(T^*), \otimes)\)-module are controlled by \( \text{End}(\text{Id}) \) in \( \text{Ext} \; (\text{Coh}(T^*), \otimes), (V_1, V_2) \).

The latter is \((\Lambda T^*_{1^*} T^*)\)-modules with \( \otimes \), so the unit is \( C \) and \( \otimes = \text{Sym} T^*_{1^*} T^* \), while \( \text{Sym} \).

For general \( G \) we get similarly \((\text{Sym} g^*) \).
Deformation of the trivial rep. Vect,$^1$

As a \((\text{Coh}(\mathcal{T}^n), \otimes)\)-module, \(\mathbb{Z}/2\) graded deforms of the 'fiber functor' Vect,$^1$ au controlled by \(\mathbb{R}\text{End}(1L)\), the tensor unit in \(\text{Ext}^1((\text{Coh}, \text{ Vect})_n, \text{ Vect}_1)\)

The latter category is \(\Lambda^*(T_1^* T^n)\)-modules with (Hopf) tensor structure over \(C\).

The tensor unit is \(C\), so \(\Rightarrow \text{Sym}(T_1 T^n) = \text{Sym}^*\).

For a general semi-simple group one can show the answer is \((\text{Sym}g_\ast)^G\).

[The degree 2 part gives \(\mathbb{Z}\)-graded deformations.]

Re-interpretation of Extra-curved Cartan G$x$

\[ \text{G} \times (A \times \text{Sym}g_\ast), \text{ dec }, W = z^a \otimes z^a (s_i) + P(z) \]

computes the (derived, non-perturbative)
(twisted) co-invariant category \(A \otimes \text{Vect}_{\mathbb{R}}\).

Example For \(T \in \mathcal{T}^n, T \neq 1\), Vector \(\otimes \text{ Vect}_{\mathbb{R}}^1 = 0\), \(T/F\)

the representations are disjoint. But after Casimir twist, \(\text{Vect}_1 \otimes \text{ Vect}_{\mathbb{R}}^1 \cong \oplus \text{ Vect} \) because \(\text{Casimir} \) has a Morse critical point on each \(\mathcal{T}_n\).
Interpretation of the BFM $T^*(G/G')$  

Recall the affine blow-up of $T^*TV/W$, also $(T^*G')^{reg}//G'$ and Spec $H^G_x(\Omega G)$ (Theorem)

$T^*_1TV/W = F$  
Lagrangian support of some $G$-category $E$ as object in $KRW$

the (nonperturbative) gauged category

Hamiltonian displacement of $F$

the underlying category

topologically twisted gauged category

(missing zero section)

most $E =$ exceptional fiber  
Spec $H^G_x(\Omega G)$

$G/G' \subset TV/W$

We can explore the entire BFM space via Hamiltonian displacements of the fiber $F$; this corresponds to computing (all) topologically twisted gaugings of $E$.

In torus case: Linear Hamiltonians suffice and we get the spectral decomposition of $E$ over $T^!$.

If $\text{Supp } E$ is not closed or has singularities, then some topologically twisted gaugings of $E$ will be ill-behaved.
A lower-dimensional analogue

This is equivariant homology/cohomology of a chain complex with infinitesimally trivialized $G$ action.

This $H^*_G(V^*)$ is a module over $H^*(BG) = (\text{Sym}g^*)^G$, so fibers naturally over $G^*/G = T/W$.

The derived fiber over $0$, $C^*(BG; V^*) \otimes C^{c^*BG}$ is quasi-isomorphic to $V^*$, the original complex.

We can put on the Kapustin goggles and double the base to $H^*_G(G)$ to get an odd version of BFM space. This is an affine blow-up of $(\text{Sym} t^* \otimes \Lambda t)^W$

Lagrangians of $H^*_G(V^*)$ are necessarily supported on $F$ then $\otimes$ (set-theoretically)

\[ \mathcal{E}/W = \text{Spec} H^*(BG) \]
Further directions

There is a Loop Group analogue of the story the Curved Caten complex and BFM space, which is roughly the blow-up of $(T^*T^*)/W$.

It should be a $2k$-dimensional reduction along the circle of 3D A-style gauge theory; it is definitely connected to $K$-theoretic GW invariants. Indeed, for a point one computes the "Verlinde" version of Witten's integration formulas.

The collection of all these theories should represent $(S^1\text{-reduced})$ 4-dimensional pure gauge theory for $G$. Langlands duality ought to be a salient feature here.

Finally, one can guess that $k$-dimensional, A-style gauge theories are "described" by $k$-categorifications of $H_k^G(\Sigma^k B G) = H_k^G(\Sigma^{k-1} G)$.

If $k = 1$, $H_1^G(\Sigma^0 B G) = H_1^G(G)$.

This gives at least a starting point in high dimension.