IV. The World through Kapustin goggles

and Rozansky's

Following Kapustin's suggestion for the construction of Rozansky-Witten theory we will now find a global interpretation of "matrix factrization" categories.

This will be consistent with the toric mirror construction and the interpretation of fixed-point categories as fibers of a map to the dual torus (and later, to conjugacy classes in the Langlands dual group).

It will also be compatible with the fixed-point construction via the Curved Cartan Complex (next lecture), and a beautiful description of $H_*^G(\Omega G)$ due to Bezrukavnikov-Finkelberg-Mirković.

A one-sentence summary of the story is that

The trivial, A-model gauged 3dim TQFT is the Rozansky-Witten theory of the BFM $T^*(G^\vee /\!/_{G^\vee}^{ad})$

or

Exhibiting a (local) A-type action of G on a category $\mathcal{C}$ is the same as exhibiting $\mathcal{C}$ as an object in the Kapustin Rozansky-Witten 2-category for $T^*(G^\vee /\!/_{G^\vee}^{ad})$.

Recollection of the problem

X smooth affine scheme, $W : X \to \mathbb{C}$ isolated crit pts.

define $MF(X;W) = \bigoplus_{\substack{\text{crit.} \\ \text{pts. } c}} MF_c(X; W - W(c))$

$$MF_c = \left\{ \left(P \underset{q}{\overset{p}{\rightleftarrows}} Q \right) \middle| \begin{array}{l} P, Q \text{ projectives defined} \\ \text{near } c, \; pq = W = qp \end{array} \right\}$$

(a dg category)

[$D(MF_c) = MF_c / \text{homotopies}$ a $\mathbb{Z}/2$ graded triangulated category]

Thm (Orlov) $D(MF_c) \cong \dfrac{D(W^{-1}(c))}{D(\mathrm{Perf}\, W^{-1}(c))}$

in particular is supported at c, and MF is supported over the critical ~~values~~ Points of W.

The Mirror construction for toric varieties would follow naturally from the assumption that

The restriction of $MF(X, W)$ to a smooth subscheme $Y \subset X$ is $MF(Y; W|_Y)$.

However, this is FALSE in the above construction, and NO VERSION of $MF(X,W)$ which is local on X will do.

Example: $Y \cap \mathrm{Crit}(W) = \emptyset$ but $W|_Y$ has critical pts.

This is fixed by Kapustin's goggles, which see T^*X.

One level down for practice

Let $\mathcal{C}$ be a 'complicated' ($\mathbb{C}$-linear) category

$a \in \mathrm{Ob}\,\mathcal{C}$ an "easy' object.

Try to study $\mathcal{C}$ by localizing at a:

Replace $x \in \mathcal{C}$ by $\mathrm{Hom}(a,x)$, right $\overset{= A}{\mathrm{End}(a)}$-module

$x \xrightarrow{f} y$ induces $f_*: \mathrm{Hom}(a,x) \to \mathrm{Hom}(a,y)$

[If this detects isomorphisms, then a is a "generator" and if $\mathcal{C}$ is abelian, cocomplete & a compact then $\mathcal{C} = A$-modules]

Try to get a different part of $\mathcal{C}$ by localizing at $b \in \mathrm{Ob}\,\mathcal{C}$, get $B = \mathrm{End}(b)$-modules.

How do we patch these together?

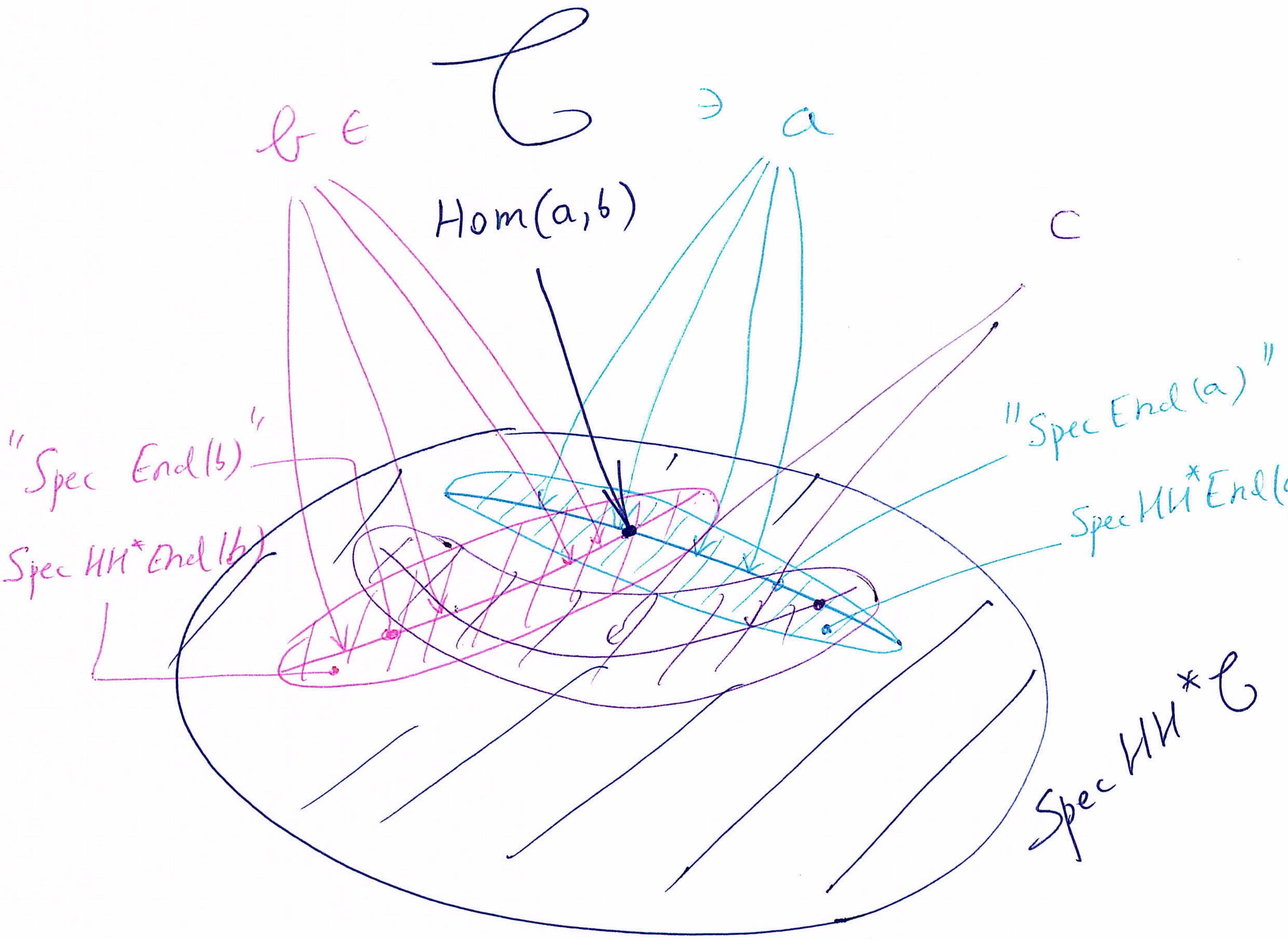
Answer: They all live over $\mathrm{Spec}\, HH^*(\mathcal{C})$, a (derived) symplectic manifold with degree 1 symplectic form.

[1-Gerstenhaber algebra, dg algebra with Poisson bracket of deg (-1)]

For each a, $\mathrm{End}(A)$ is an algebra over $HH^*(\mathcal{C})$, with co-isotropic support. [Lagrangian?]

In lucky cases, $HH^*\,\mathrm{End}(A)$ is a local chart in $\mathrm{Spec}\, HH^*\mathcal{C}$.

Pictorially, with Kapustin's goggles



Note the double vision caused by the goggles:

$$\mathrm{End}(a)^\circ \underset{HH^*\mathcal{C}}{\otimes} \mathrm{End}(b) = \mathrm{End}\big(\mathrm{Hom}(a,b)\big)$$

(subject to some finiteness of a and b)

Hochschild Cohomologies

An object of categorical depth $k \geq 0$ has a succession of $(k+1)$ Hochschild cohomologies, ranging in categorical depth from k to 0, but with an additional algebra structure of commutativity ranging from 0 to k (E_1 to E_{k+1} algebra objects)

Examples	k	Hochschild stuff	
		((-1) HH* dim V)	
Vector space V	0	$0\text{-}HH^* = \mathrm{End}(V)$	E_1 algebra (vect sp.)
Algebra or category	1	$0\text{-}HH^* = A \otimes A^\circ\text{-mods}$ $\Downarrow$	E_1 category
		$1\text{-}HH^* = \mathrm{End}(\mathrm{Id})$	E_2 vector sp.
		$= \mathrm{Ext}_{A\otimes A^\circ}(A,A)$	
2-algebra or 2-category $\mathcal{C}$	2	$0\text{-}HH^* = \mathrm{Funct}(\mathcal{C},\mathcal{C})$ $\Downarrow$	E_1 2-category
		$1\text{-}HH^* = \mathrm{End}(\mathrm{Id})$ $\Downarrow$	E_2 category
		$2\text{-}HH^* = \mathrm{End}(\mathrm{Id})$	E_3 vector space [algebra]

Eg if A is an E_2 algebra, A is an algebra over itself

$1\text{-}HH^*$ is $A \otimes_{A\otimes A^\circ} A^\circ$-modules

$2\text{-}HH^*$ is $\boxed{\mathrm{Ext}_{A\otimes_{A\otimes A^\circ} A}(A,A)}$

Point of Hochschild Cohomologies

is deformation theory:

The top (k^{th}) Hochschild cohomology of a k-categorical object controls its formal deformation theory via the Maurer-Cartan equation "$d\gamma + \frac{1}{2}[\gamma,\gamma] = 0$".

This is an act of faith, but obvious for $k=0$, "classical" for $k=1$, proved by J. Francis for general k, but special k-categories (E_k type)

Examples

- $k=0$, $V^\cdot$ a complex, $0\text{-}HH^* = End(V^\cdot)$, ∞-mal deformations $\gamma \in End^1(V^\cdot, d)$, such that $(d+\gamma)^2 = 0$.
- $k=1$, $A^\cdot$ a dga, cochains for $1\text{-}HH^2$ are maps $\mu: A^{\otimes 2} \to A$ of degree 0, Maurer-Cartan $\Leftrightarrow$
orig. mult $m+\mu$ is an associative multiplication.
[Actually, $1\text{-}HH^2$ contains cochains $m_p: A^{\otimes p} \to A$ of degree $2-p$, giving A_∞ deformations].

Kapustin's goggles for 2-categories

The 2-categorical picture is identical, except:

- $2\text{-}HH^*(\mathcal{C})$ replaces $1\text{-}HH^*$;

  its Spec is a (dg) symplectic manifold, $\deg \omega = 2$

- $1\text{-}HH^*(a)$ $\left(= End_{End(a)}(Id)\right)$ replaces $End(a)$

Key Example

- X a smooth complex manifold
- $\mathcal{C} = (\mathcal{O}_X, \otimes)\text{-mod} = 2cat$ of $\mathcal{O}_X$-linear categories
- $\mathrm{Spec}\ 2\text{-}HH^*(\mathcal{C}) = \hat{T}^*X$ with fiber of deg. -2
- $a = Coh(X)$: $1\text{-}HH^* = Ext_{\mathcal{O}_X}(\mathcal{O}_x, \mathcal{O}_x) = \mathcal{O}_X$,

  sitting on the zero section

- $b = Vect_x$; $1\text{-}HH^* = Ext_{\Lambda^\cdot T_x}(\mathbb{C}, \mathbb{C}) = Sym\, T_x$,

  functions on the fiber T_x^*

- $c = MF(X, W)$: $1\text{-}HH^* = \mathcal{O}_{\Gamma(dW)}$, lives on

  the graph of dW! (assume isolated singularities)

Remark: $\cancel{Ext}$ $Tor_{T^*X}\left(\mathcal{O}_X^{op}, \mathcal{O}_{\Gamma(dW)}\right) = \mathcal{O}/\langle \partial_i W\rangle \cong HH^*(MF(X,W))$

"double vision"

Choucroûte & Choucroûte garnie of Y

Kapustin conjectured that any holomorphic symplectic manifold Y should carry a natural sheaf of 2-categories, of which Y is the 2-HH*. ($\mathbb{Z}/2$ graded).
He posited objects for smooth Lagrangians and described the Hom's for nice intersections.

Here is a proposed implementation CG(Y) (categorified Y) of this, which could also be called the choucroûte garnie of Y (the choucroûte).

The germ of CG(Y) near any smooth Lagrangian L is the 2-cat of $\mathcal{O}_L$-linear categories.*

Fact: Formally, this is a consistent definition: Hamiltonian motions of L lift to this sheaf of 2 cats

Missing: Convergent deformation theory to produce genuine germs that can be patched.

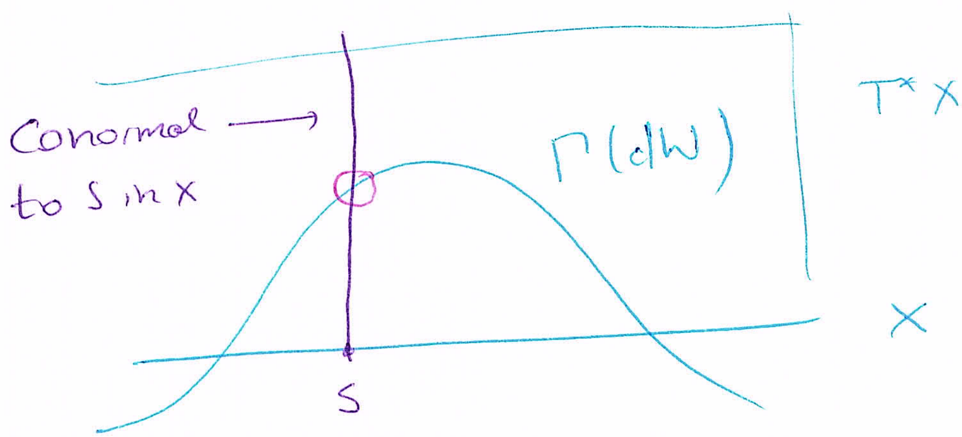
Assuming this works: Any "potential" W on X defines a natural object in $CG(T^*X)$, supported on $\Gamma(dW)$. $\mathrm{Hom}(ob_{X,0}, ob_{X,W}) = MF(X,W)$.

* after choice of Darboux coordinates

Mystery of restriction to submanifolds explained:

Even if $W: X \to \mathbb{C}$ has no critical points meeting $S \subset X$, $\Gamma(dW)$ will intersect the conormal bundle to S in X precisely over the critical points of $W|_S$.

The Hom categories between the two in the Chouchoüt garnie of T^*X are the $MF(S; W|_S)$.



Torus fixed points revisited

A ($\mathbb{C}$-linear) category $\mathcal{C}$ with locally trivialized T-action determines the formal germ along the zero-section $T^\vee$ of an object $\hat{\mathcal{C}}$ in $CG(T^*T^\vee)$.

The underlying category $\mathcal{C}$ is $\mathcal{H}om_{CG}(\text{Zero}, \hat{\mathcal{C}})$

$$\left(= \mathcal{H}om_{T/T^{loc}}(\text{Regular rep}, \mathcal{C})\right)$$

The fixed point category $\mathcal{C}^{T/T^{loc}}$ is $\mathcal{H}om_{CG}(\widehat{T_1^*T}, \hat{\mathcal{C}})$

$$\left(= \mathcal{H}om_{T/T^{loc}}(\text{Triv. rep}, \mathcal{C})\right).$$

Clearly, for ~~the~~ good behaviour we would want $\hat{\mathcal{C}}$ to be enhanced to a "reasonably finite" object in $CG(T^*T^\vee)$, eg one with proper support; so an uncompletion (delocalization) of $\hat{\mathcal{C}}$ is desirable.

Recall that a mirror of a Fukaya $\mathcal{C}$ can have the form (Y, W) with a holomorphic map $\phi: Y \to T^\vee$.

~~It's supp~~ The graph of $d\phi$ in $T^*Y \times T^*T^\vee$ gives a "functor" $CG(T^*Y) \to CG(T^*T^\vee)$, and the requisite properness is that of $\mathrm{Crit}(W)$ over $T^\vee$.

The Bezrukavnikov-Finkelberg-Mirković base

Theorem [BFM] G reductive connected, $G^\vee$ the Langlands dual group, ΩG the based loop group.
Then $\mathrm{Spec}\, H_*^{G_{ad}}(\Omega G;\mathbb{C}) = (T^*G^\vee)^{reg}/\!\!/_{ad}\, G^\vee$
and is the Weyl group quotient of the "affine blow-up" of $T^*T^\vee$ arising by adjoining $\frac{e^\alpha - 1}{\alpha}$ (roots α in $G^\vee$ in $(\mathfrak{g}^\vee)^*$)

Example ($G = SL_2$, $G^\vee = PSL_2$)