

Example from algebra - Group actions continued
① of a locally trivialized trans action

\mathcal{C} = category of finite-dim reps of $t = \text{Lie}(T)$

sheafifies over its spectrum t^* (set of eigenvalues)

Specifically: equivalent to finitely supported
colimit sheaves on t^* .

t^* maps to T^* so we get a $\text{Coh}(T^*)$ -action
which comes from the following action of $T/\exp(t)$:

- For each $t \in T$, $F_t : \mathcal{C} \rightarrow \mathcal{C}$ is Id .
- For $t = \exp(\xi)$, the trivialization of $F_t = \text{Id}$
is the automorphism $\exp(\xi)$ on
each representation.

So the fibration to T^* is the analogue of the
special decomposition of a vector space under commuting
operators.

Remark You might think that the spectrum
should be t^* and not T^* , but this is not so.

That's because, for $\phi \in \mathcal{C}^*: T \rightarrow \mathbb{C}^*$, tensoring with
 ϕ_u is an automorphism of \mathcal{C} which commutes with
the ^{locally trivialized} group action:



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$$F_t \circ (\mathbb{C}_\mu \otimes \cdot) \xrightarrow{a_t = \text{circular arrow}} (\mathbb{C}_\mu \otimes \cdot) \circ F_t$$

$$s \downarrow e^{2+\mu} \qquad s \downarrow e^2$$

$$\text{Id} \circ (\mathbb{C}_\mu \otimes \cdot) \stackrel{(1)}{=} (\mathbb{C}_\mu \otimes \cdot) \circ \text{Id}$$

Moral of the story

For vector spaces, reps of T have a spectral decomposition into 1-dim. representations, classified by points in the Pontryagin dual group $H^2(BT; \mathbb{O}^\times)$.
 $= H^2(BT; \mathbb{Z})$.

For categories, reps of T have a spectral decomposition (over $\text{Coh}(T^\vee)$) parametrized by

$$\text{the Langlands dual torus } T_C^\vee = H^2(B(T/\mathbb{F}_{\text{loc}}); \mathbb{O}^\times) \\ = H^2(BT; \mathbb{Q}^\times).$$

Recall that this last group classifies the actions of T/\mathbb{F} on Vect, so we have again decomposed the category into isotypicals.

"The categorified Pontryagin dual of "Theoc"
is T_C^\vee !"

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The fixed-point category - geometric interpretation (1st go)

The "skyscraper category" Vect_1 at $1 \in T^\vee$ is the trivial representation of T/T_{loc} .

[a sheaf $\mathcal{E} \in \text{Coh}(T^\vee)$ acts on Vect_1 by tensoring with \mathcal{E}_1 , the fiber at 1]

So it is reasonable to guess that the fixed-point category $\mathcal{E}^{T/T_{\text{loc}}}$ is (a version of) "the fiber of \mathcal{E} over $1 \in T^\vee$ ".

For instance, if $\mathcal{E} = \text{Coh}(Y)$ with $Y \rightarrow T^\vee$, then $\mathcal{E}^{T/T_{\text{loc}}} = \text{Coh}(Y_1)$, subsheaves supported on the scheme-theoretic fiber over 1.

[A derived version also exists; see work of Ben-Zvi, Francis, Nadler for defining $\mathcal{E} \otimes \text{Vect}_1$,
 $\text{Hom}_{\text{Coh}(T^\vee)}(\text{Vect}_1, \mathcal{E})$ and derived versions]

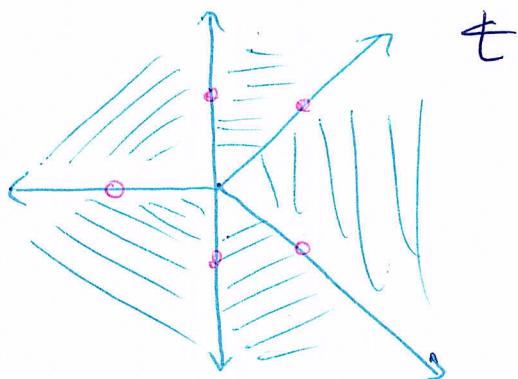
The guess is essentially correct with a huge proviso, enfraged by physicists!

Actually, fixed by collapse of grading mod 2.

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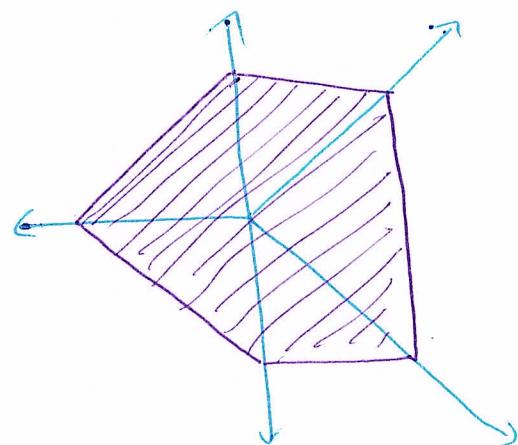
Toric Varieties and their minors

A toric variety for $T = (\mathbb{C}^\times)^n$ is determined by its (rational polyhedral) fan, a collection of ^{convex} rational polyhedral cones with disjoint interiors and meeting along faces, all in $\mathfrak{t} = \text{Lie}(T)$.



Complete fan and primitive vectors on rays
They correspond to the Weil divisors; toric

- Each cone in the fan



Convex polytope strictly compatible with fan
[The fan can be recovered from the polytope]

determines a monoid inside the lattice of characters M of T , and Spec of the corresponding ring is an affine chart. Glue along faces.

Properties:

- Fan cones $\mathfrak{t} \Leftrightarrow$ toric var. is proper
- Fan is integrally simplicial \Leftrightarrow smooth
- Strictly compatible rational polytopes are related to ample line bundles

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Polytype and dual polytype

More precisely, ample line bundles correspond to
 strictly convex, integer-valued, piecewise linear functions φ
 on the (support of the) fan and a polytype P_φ
 defined by $\varphi(z) \leq 1$.

$$= P_\varphi^*$$

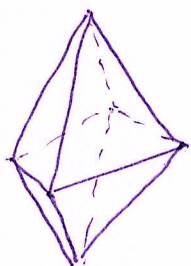
The polar polytype $\{n \in t^* \mid n(P_\varphi) \leq 1\}$ is
 the moment map image of X for the torus action
 in the polarization defined by φ .

The variety X itself is the symplectic (GIT) quotient
 of \mathbb{C}^R ($R = \text{set of rays}$) under the torus K

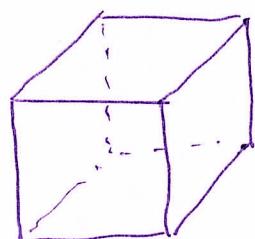
kernel of $1 \rightarrow K \rightarrow (\mathbb{C}^*)^R \xrightarrow{T} T \rightarrow 1$,

primitive
vector

(with the standard symplectic form $\sum dz_i \wedge d\bar{z}_i$
 and linearization defined from φ).



polytype



polar dual polytype

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The Landau-Ginzburg Mirror of X

is the pair (X^\vee, w) where

- $X^\vee \hookrightarrow T^\vee$, the dual torus to T
- $= T_{\text{cpt}}^\vee \times \exp(P_q^*)$ image of moment map.
- $w = \sum g_j e^{p_j}$

where the p_j are the primitive rays and the $g_j \in \mathbb{C}^\times$ have modulus $\exp(\varphi(p_j))$ and phase representing a B -field on X .

[Fact: $0 \rightarrow M = \text{Hom}(T, \mathbb{C}^\times) \rightarrow \mathbb{Z}R \rightarrow H^2(X, \mathbb{Z}) \rightarrow 0$]
if X complete

Remark: In algebraic geometry one takes $X^\vee = T^\vee$. There are reasons to restrict to X^\vee , but the reasons smell of stability (à la GIT).

Part of the claim is that the Fukaya category of X is equivalent to the Matrix Factorization category of (X^\vee, w) .

Since we have a natural map to T^\vee , hence a natural $(\text{Coh}(T^\vee), \otimes)$ -module structure, this also captures the action of T_{cpt} on the Fukaya cat!

Derivation from the mirror of \mathbb{C}^R

Using the "ansatz" for the fixed-point category, the mirror of X can be deduced solely from knowledge of the mirror of \mathbb{C}^R as the torus $((\mathbb{C}^\times)^R)^\vee$ with potential $W = \sum z_i$.

The action of $(\mathbb{C}^\times)^R$ on the Fukaya cat. is encoded by the projection to $((\mathbb{C}^\times)^R)^\vee$, that of K by the projection to K^\vee .

The fixed-point category is the fiber over $1 \in K^\vee$, which is T^\vee with the previous W .

Note: Left out the Novikov coefficients q_i

Note: Restricting to the moment map image has the virtue of making the fiber empty when the GIT quotient X is so. (Virtue or vice?)

Note: ~~theories~~

This was the original derivation of the mirror construction (Givental, Mori-Yafa) of course without mention of categories..

Matrix Factorizations

(O2Lor)

X smooth manifold, $w: X \rightarrow \mathbb{C}$ holomorphic function

~~A matrix factor~~

Choose a unit pt $\infty \rightarrow$ shift w to make $w(\infty) = 0$

A matrix factorization is

$$P \xrightleftharpoons[2]{\Phi} Q$$

P, Q projection modules
 \mathcal{O}_X

\mathcal{O} -linear maps

such that $\Phi Q, Q \Phi = w$.

$\text{Hom}(P_1 \rightarrow Q_1, P_2 \rightarrow Q_2)$ is a complex

Def Derived cat of matrix factorizations

Object: \uparrow , $\text{Hom}:$ $\frac{\text{original Hom}}{\text{homotopies}}$.

$$\text{D} \frac{\text{Coh}(w^\vee(\mathcal{O}))}{\text{D} \text{Perf}(w^\vee(\mathcal{O}))}$$

Theorem $\text{D}(\text{MF}_*(X, w)) \cong$

Prop: this is supported at ∞ .

Notice: $\mathbb{Z}/2$ graded.

Context:

$HH^*(O_X)$ controls the deformation theory
of O_X as an association

HH^3 Algebra
obstructions

$HH^2(O_X)$ controls algebra deformations.

$HH^1(O_X)$: derivations

all of HH^{ev} should give deformations

all of HH^{odd} should give (generalized) derivations.

From HH^n cocycle $\rightarrow m_n: \overset{\text{con}}{A} \rightarrow (A)$
ought to have
degree $2-n$

all even m_n ($A \otimes$ multiplication)

are allowed if grading collapses mod 2!

$m_0: \mathbb{C} \rightarrow A$ or $w \in A$

Cough condition: w is central.

Deform the category of A -modules

brutally after collapsing
grading mod 2.

'Good kid' example

$$A^\cdot = (\Omega^{\cdot, \circ}_X, d)$$

$$w \in \Omega^2_d(X) \quad dw = 0$$

Can form curved modules over A^\cdot

graded vector spaces $\bigoplus M^\bullet \xrightarrow{d}$

- $\nabla^\omega = w \in A^\cdot$, A commutation
so natural left
so $HK^\omega(A)$.
- $(\Omega^{\cdot, \circ}_X, d)$ -module
- perfect (in thick closure
of shifts of $(\Omega^{\cdot, \circ}_X, d)$) .

The category depends characteristically on w !

scaling w changes the category

$$\text{If } w = g(\mathcal{L})$$

then $\otimes \mathcal{L}$: (uncurved modules) \rightarrow (curved modules)

(Kapustin's version of Rozansky-Witten category)

X is an algebra over $\text{Spec } \text{HH}^*(X)$

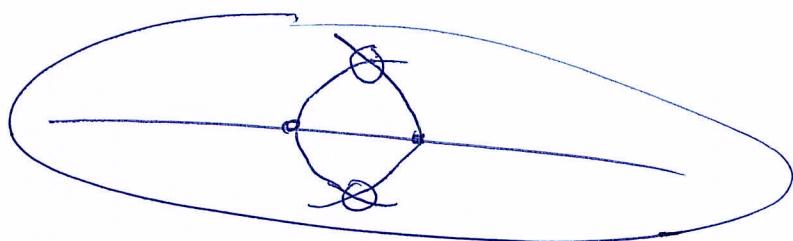
$\text{Spec } (\mathcal{O}_X \otimes \Lambda^T X)^\wedge$ as algebra
 \wedge Poisson bracket of deg -1
 \leftarrow deg 1 symplectic form
 $\pi T^* X$

$w = w_L$

Spec $\text{HH}^*(X)$

$\text{Ext}_0(k_{x_1}^{lf}, k_x^{lf})$
as algebras over
 $\text{HH}^*(X)$

$\text{Ext}_0(k_1, k_2)$
as algebras over
 $\text{HH}^*(X)$



Example

- One category level down
 - algebras, modules
- Supermanifolds needed.
 k_x skyscraper 0-moduli.

$\xrightarrow{\quad \cdot \quad}$ X

Deformation? $\mathbb{R}\mathrm{Hom}_{\mathcal{O}_X}^1(k_x, k_x)$
controls deformations

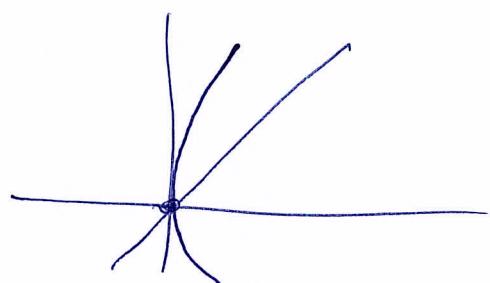
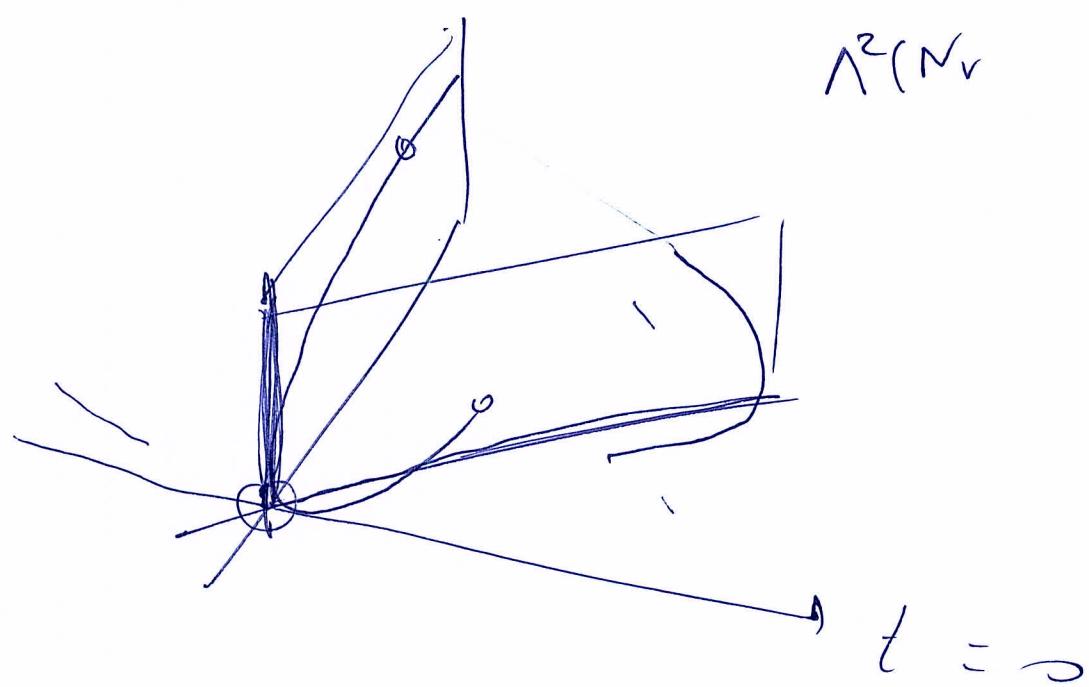
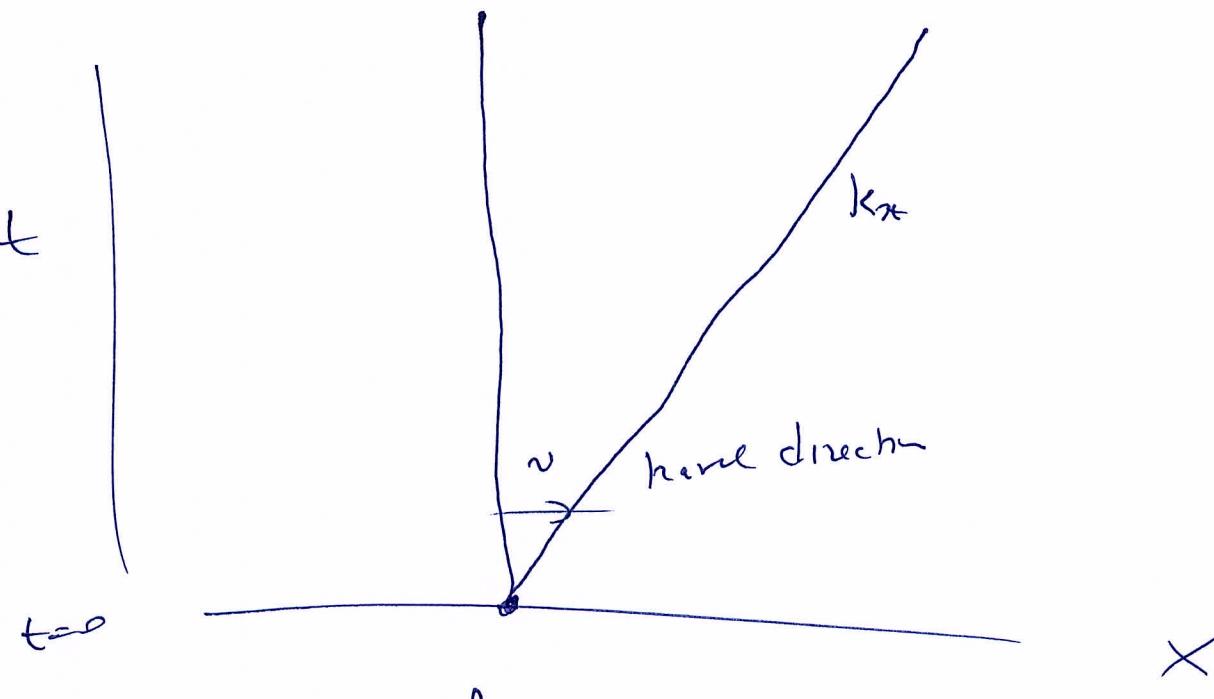
$$\mathbb{R}\mathrm{Hom}^\circ \simeq \Lambda^* T_x \quad \text{so } T_x \text{ controls def}$$

Eg, if $\dim T_x = 3$, $\omega \in \Lambda^3 T_x$ should give
 a deformation of k .

$$\begin{array}{ccccccc} \mathcal{O}_X \otimes \Lambda^3 T_x^* & \xrightarrow{\quad \cdot \quad} & \mathcal{O}_X \otimes \Lambda^2 T_x^* & \xrightarrow{\quad \cdot \quad} & \mathcal{O}_X \otimes T_x^* & \xrightarrow{\quad \cdot \quad} & \mathcal{O}_X \rightarrow k_x \\ \downarrow \omega & & \downarrow \omega & & \downarrow \omega & & \downarrow \omega \\ & & \text{vol} & & & & \end{array}$$

$\frac{\text{vol}}{k_x}$: deformation of k_x

$$\mathbb{R}\mathrm{Hom}(k_x, \frac{\text{vol}}{k_x}) \quad (\Lambda^* T_x, \omega^{-1})$$



the sum of the
parabolas is zero
at time 0

$$\text{Ext}(k_x^{\text{more}}, k_x^{\text{less}}) = \Lambda^2 N$$

t acts naturally.