Lie group actions on categories, II

The story of Lie group actions on categories branches into two major directions.

[Note: beyond the obvious distinctions of continuous vs smooth, algebraic, etc]

A-model actions — related to group actions on symplectic manifolds and their Fukaya categories

B-model actions — algebraic group actions on coherent sheaves on a variety

The A-actions in turn will have two variants, reflecting roughly the distinction between D-modules (analysis) and flat vector bundles (topology).

I will focus on the latter (easier, and applies to Fukaya categories).

B-actions are easier to describe, but in fact I only know a mirror story in the torus case.

Example from vector spaces

B-model G-action on a vector space:

ordinary (continuous, diff, analytic etc) linear action

For compact (or complex reductive) Lie groups, the category of reps is semi-simple, so there is no "hidden" homotopical information. Complexes with G-action will split.

A-model G-action

This is an *infinitesimally trivial* or a *locally trivial* action

That means that G acts on V, and

- the Lie algebra action is trivial (infinitesimal)
- the $\mathfrak{g}$ neighborhood of $1 \in G$ acts trivially.

Either way, within reason, the component G_1 acts trivially and the action factors through $\pi_0 G$.

But now, there is homotopical information!

That's because there is a notion of "homotopy trivialization" on a complex: $(V^\bullet, \partial)$

- G acts on $V^\bullet$, commuting with ∂
- the $\mathfrak{g}$-action is homotopy trivialized: $L_\xi = [\partial, \iota_\xi]$

$L_\xi \in End^0(V^\bullet)$ ∞-mal action

$\iota_\xi \in End^{-1}(V^\bullet)$ htpy trivialization

$[\iota_\xi, \iota_\eta] = 0.$

Example X smooth manifold

G acts on X

Let $(V^{\cdot}, \partial) = (\Omega^{\cdot}_X, d)$.

Then · G acts

· $\mathfrak{g}$ acts by the Lie derivative

· Define $\iota_\xi : \Omega^k \to \Omega^{k-1}$

= contraction with

the vector field

Cartan formula: $L_\xi = [d, \iota_\xi]$

Clear that $[\iota_\xi, \iota_\eta] = 0$.

$\overset{\shortparallel}{\iota_\xi \iota_\eta + \iota_\eta \iota_\xi}$

$\Longrightarrow$ define equivariant cohomology of X!

$$\begin{array}{ccc} X \hookrightarrow & X_G & = (EG \times X)/G \\ & \downarrow & \\ & BG & \end{array}$$

$H^*_G(X) = H^*(X_G)$, $H^*(BG)$-module,

$\shortparallel$ Cohomology of BG with twisted coeffs in $H^*(X)$

twisted!

G-action on $H^*(X)$ is trivial

But G action on (Ω^*_X, d) is only

homotopically infinitesimally trivial.

Eg $X = G$, translation action

X_G = contractible, $H^*(X_G) = \mathbb{C} \neq H^*(BG; H^*(X))$ $\Longleftarrow$ $\|$ E_2 term of spectral sequence

The Cartan model for equivariant cohomology:

$$\left[(\Omega^\cdot_X, d) \otimes \mathrm{Sym}\, \underbrace{\mathfrak{g}^*}_{\text{degree } 2} \right]^G$$

$$d \rightsquigarrow \underbrace{d + \xi^a \cdot \iota_a}_{d_c}$$

ξ_a basis of $\mathfrak{g}$

ξ^a dual basis of $\mathfrak{g}^*$

Prop (1) $d_c^2 = 0$ on the invariant part

$$\left(= \xi^a \cdot \mathcal{L}_{\xi_a} \right)$$

(2) Cohomology $\cong H^*_G(X)$

Eg $H^*(BG) = (\mathrm{Sym}\, \mathfrak{g}^*)^G$

Precise story: model for EG = princ G-bundle on BG from $\cdot \wedge \mathfrak{g}^*$

from $\mathrm{Sym}\, \mathfrak{g}^*$ $\nearrow$ BG

("Weil model for equivariant cohomology)

~~Example~~

Comment explaining the relation:

$$\begin{tikzcd}
\mathfrak{g} & = & \mathfrak{g} \arrow[r, "L_x"] & \mathrm{End}^0(V,\partial) \\
\arrow[u] & & \arrow[u, "\partial = \mathrm{Id}"'] & \arrow[u, "{[\partial, \cdot]}"'] \\
0 & -1 & \mathfrak{g} \arrow[r, "\iota_x"] & \mathrm{End}^{-1}(V,\partial)
\end{tikzcd}$$

dg Lie algebra

Lie algebra homomorphism

$$\begin{cases} \mathfrak{g}_0 : \text{ usual relation} \\ \mathfrak{g}_{-1} : \text{ abelian} \\ \mathfrak{g}_0 \text{ acts on } \mathfrak{g}_{-1} \text{ naturally.} \end{cases}$$

"Factoring $\mathfrak{g} \to \mathrm{End}(V)$ via the zero Lie algebra is a homotopy trivialization".

$$\text{zero Lie algebra} = \begin{array}{c} \mathfrak{g}^{(0)} \\ \uparrow \scriptstyle{\mathrm{Id}} \\ \mathfrak{g}^{(-1)} \end{array}.$$

Definition

A B-model action of G on a category is an action as a group which preserves the structure you want

"Placing a structure on something"

= realizing it as a sheaf over some Grothendieck site.

X smooth mfld $\xrightarrow{\text{defines}}$ functor on the cat of smooth mfolds

$$Y \longmapsto C^\infty(Y, X)$$

[this is a sheaf in an appropriate topology]

$\mathcal{C}$

Giving a smooth structure to $\mathcal{C}$

= enhancing it to a sheaf ~~functor~~ on the (Grothendieck site) of smooth mfolds

G is a sheaf of groups on the same

structure-preserving action of G on $\mathcal{C}$

= action of the group sheaf on sheaf of categories.

[Joyal, Tierney - "model structure" on sheaves of categs over a Grothendieck site]

A-model action

- G acts on the category $\mathcal{C}$ (structure-preserving)

structure: at least differentiable

want: the action of $\hat{G}$ (= formal group at 1) has been trivialized (homotopy trivialized)!

Example G acts on $U\mathfrak{g}$ by conjugation

(~~action~~ induces an action on the category of $\mathfrak{g}$-modules)

Obs The action of $\mathfrak{g}$ is inner:

L_ξ acts by $[\xi, .]$

$\xi \in U\mathfrak{g}$.

Exercise This gives a natural iso between the ~~action~~ induced action of $\mathfrak{g}$ on ($U\mathfrak{g}$-mod) and the trivial action!

If $h \in \hat{G}$, ~~the rep~~ V ~~a rep of~~ $\in U\mathfrak{g}$-mod

then $h(V)$: $u \in U\mathfrak{g} \mapsto \rho_V(h u h^{-1})$.

If h acts "internally" on V by $\rho_V(h)$,

then $V \to h(V)$

Definition

Preamble G acts on $\mathcal{C}$ (differentially)

Then $\mathfrak{g}$ "acts on $\mathcal{C}$"

$$\xrightarrow[\text{Lie alg map.}]{} HZ^1(\mathcal{C}) \quad \text{Hochschild 1-cocycles.}$$

Recall that HH^* is a Lie algebra (after deg shift by 1)

HH^1 : Derivations / Inner derivation.

A trivialization of the ∞-mal action is a factorization ~~thro~~ through the 0 Lie algebra

$$\begin{array}{ccc} \mathfrak{g} = \mathfrak{g}_0 & \xrightarrow{L_\xi} & HZ^1(\mathcal{C}) \\ \| \uparrow \partial & & \uparrow \beta \quad \text{Hochschild differential} \\ \mathfrak{g}_{-1} & \xrightarrow{\iota_\xi} & HC^0(\mathcal{C}) \end{array}$$

- This is a map of dgLas
- G-equivariant.

In particular, $[\iota_\xi, \iota_\eta] = 0$.

Eg $\mathcal{C} = A$-module, $HH^*(\mathcal{C}) \cong HCH^*(A)$

$$A \to \mathrm{Hom}(A, A) \to$$
$$a \mapsto [a, \cdot]$$

Example X manifold, G action,

$(\Omega^\cdot_X, d)$ dga ∞-mod

The same formulas give a trivialization of the action as $\overset{dg}{}$algebra action.

Guess? ~~Quotient category should~~ Fixed pt category =

$$\left((\Omega_X \otimes \mathrm{Sym}\, \mathfrak{g}^*)^G, d_c \right)\text{-modules.}$$

<u>False</u> (Related to twisted sectors)

a correction exists (curved Cartan complex).

A locally trivialized action

G-action on a category

trivialized on $U \ni 1$ nbhd in G.

"

have natural isomorphism

$$F_g \xrightarrow[\varphi_g]{\sim} \mathrm{Id} \qquad \text{for all } g \in U$$

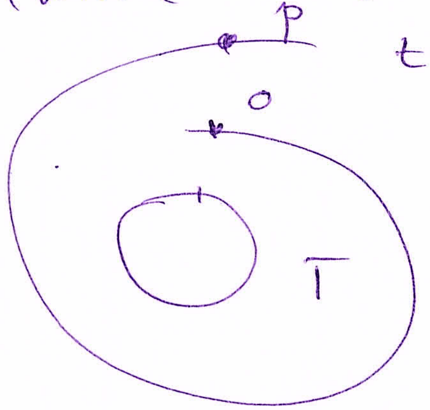
Coherent under multiplication whenever mult. is defined.

Case of a torus

Have T acting locally trivially on $\mathcal{C}$.

Have $\exp: \mathfrak{t} \longrightarrow T$ covering, kernel $\pi = \pi_1 T$.

<u>Claim</u> The trivialization of the action can be continued coherently to all of $\mathfrak{t}$.
(usual analytic cont. argument)



action of ~~exp~~ $p \in \mathfrak{t}$ is trivial for <u>two reasons</u>:

(1) It maps to $1 \in T$

(2) Have trivialized the action on $\mathfrak{t}$.

Get $p \longrightarrow \mathrm{Aut}(\mathrm{Id}_{\mathcal{C}})$ group homomorphism

$$\mathbb{C}[\pi] \xrightarrow{\text{ring hom}} \mathrm{End}(\mathcal{I}_{\mathcal{C}}) = HH^0(\mathcal{C})$$

$\Vert$ group ring of T [E_2 homomorphism]

Get a stratification of $\mathcal{C}$ over $\mathrm{Spec}\, \mathbb{C}[\pi] = T^\vee$

Action of T/T_{loc} on $\mathcal{C}$ is captured by a ~~fib~~ fibration of $\mathcal{C}$ over $T^\vee$

$\Updownarrow$

$\mathcal{C}$ becomes a module category over

$$\left(Coh(T^\vee), \otimes \right).$$

Ex If $\mathcal{C} \cong Coh(Y)$

then a map $Y \to T^\vee$ would provide such a structure.

~~E~~ X: compact symplectic manifold

given a action of T ~~acts~~ on X ~~by~~ Hamiltonian

then magic T acts locally trivially on the Fukaya cat.

If X has a mirror Y,

expect: $Y \longrightarrow T^\vee$ holomorphic

this captures the action of T on Fukaya(X)!