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Note Title 3/11/2009

$\mathcal{A}$ abelian category $\rightsquigarrow$ Hall algebra $\mathcal{H}_\mathcal{A}$
assoc. algebra $/\mathbb{C}$
(e.g. quantum groups, ring of symmetric functions, ...)
- construction gives natural bases!

Strong finiteness assumption on $\mathcal{A}$:
$|\mathrm{Hom}_\mathcal{A}(M,N)| < \infty$

e.g. $\mathcal{A} = \mathrm{Vec}_k$, $|k| < \infty$
$\mathcal{A} = \mathrm{Mod}(R)$ R f.d. alg$/k$, $|k|<\infty$
$\mathcal{A} = \mathrm{Coh}(X)$, X smooth projective$/k$, $|k|<\infty$

$$\mathcal{X}_\mathcal{A} := \mathrm{Ob}(\mathcal{A})/\mathrm{isom}$$

$$\mathcal{H}_\mathcal{A} = \{ f : \mathcal{X}_\mathcal{A} \to \mathbb{C} : |\mathrm{Supp}\, f| < \infty \}$$
as vector space

Convolution product:

$$f * g(M) = \sum_{N \subseteq M} f(N)\, g(M/N)$$

... finite sum by our strong assumption.

Claim $(\mathcal{H}_\mathcal{A}, *)$ is an associative algebra with unit 1_0 = char. function of the 0 object.

Pf. $(f_1 * f_2) * f_3 (M) =$

$$= \sum_{N \subseteq M} f_1 * f_2 (N) \cdot f_3 (M/N)$$

$$= \sum_{P \subseteq N \subseteq M} f_1(P)\, f_2(N/P)\, f_3(M/N)$$

$$f_1 * (f_2 * f_3)(M) = \sum_{P \subseteq M} f_1(P)\, f_2 * f_3 (M/P)$$

$$= \sum_{P \subseteq N \subseteq M} f_1(P)\, f_2(N/P)\, f_3(M/N)$$

Remark $f_1 * \dots * f_n (M) = \sum_{0 = M_0 \subset M_1 \subset \dots \subset M_n = M} f_1(M_1/M_0) \dots f_n(M_n/M_{n-1})$

Example $\mathcal{A} = Vec_k$

$$\mathcal{X}_{\mathcal{A}} \simeq \mathbb{N}$$

$1_n := [k^n] \longleftarrow n$

$$1_m * 1_n (k^p) = \sum_{U \subset k^p} 1_m(U)\, 1_n(k^p/U)$$

$$= \begin{cases} 0 & p \neq m+n \\ |Gr_m(k^p)|, & p = m+n \end{cases}$$

Recall q-numbers $\quad [n] = \dfrac{q^n - 1}{q-1} \qquad q = |k|$

$[n]! = [n][n-1]\dots[1]$

$$\rightsquigarrow \quad |Gr_m(k^p)| = \begin{bmatrix} p \\ m \end{bmatrix} = \frac{[p]!}{[m]!\,[p-m]!}$$

So $\quad 1_m * 1_n = \begin{bmatrix} m+n \\ m \end{bmatrix} \cdot 1_{m+n}$

$$\rightsquigarrow \quad \mathcal{H}_{\mathcal{A}} \simeq \mathbb{C}[x]$$

$$\begin{array}{ccc} \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ 1_n [n]! & \longleftarrow\!\!\shortmid & x^n \end{array}$$

Example 2

$\mathcal{A} = \{ (V, x) : V \text{ f.d.}/k, \ x \in \operatorname{End} V \text{ nilpotent} \}$

$= \ Coh_0(\mathbb{A}^1(k))$ $\quad$ coherent sheaves on $\mathbb{A}^1$ supported at 0

$= \ Rep_k^{nil}(\circlearrowright)$ $\quad$ nilpotent reps of quiver

$\mathcal{X}_{\mathcal{A}} \simeq \{ \text{partitions} \}$ $\quad$ (Jordan normal form)

$= \{ \lambda = (\lambda_1, \lambda_2, \dots \lambda_n, 0, \dots 0) \}$

$\lambda_1 \geq \lambda_2 \geq \dots \lambda_n$

$(k, 0) \longleftrightarrow (1)$

Let's compute a product here, as a point-count:

$1_{(1)} * 1_{(1)}((V,x)) = 0$ unless $\dim V = 2$

$1_{(1)} * 1_{(1)}(k^2, 0) = |\mathbb{P}^1(k)| = q+1$

$1_{(1)} * 1_{(1)}\left(k^2, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = |k^*| = q-1$

$\Rightarrow 1_{(1)} * 1_{(1)} = (q+1) 1_{(1^2)} + (q-1) 1_{(2)}$

In fact $\mathcal{H}_{\mathcal{A}} = \mathbb{C}[1_{(1^r)}]_{r \geq 0}$

$\left((1^r) \leftrightarrow (k^r, 0)\right) \simeq \mathbb{C}[x, \ldots]^{\sigma_\infty}$

ring of symmetric functions in ∞ many variables ... connection to Hall-Littlewood polynomials - depend on q, interpolate between monomial & Schur symmetric functions. $1_{(1^r)}$ go to these symmetric functions, as a function of $q = |k|$.

Further structure on $\mathcal{H}_{\mathcal{A}}$:

- Green's inner product

$$(f,g) = \sum_{M \in \mathcal{X}_{\mathcal{A}}} \frac{f(M)\, g(M)}{a_M} \qquad a_M = |\mathrm{Aut}\, M|$$

$\rightsquigarrow$ topological coproduct

$$\Delta : \mathcal{H}_{\mathcal{A}} \longrightarrow \mathrm{Fun}\left(\mathcal{X}_{\mathcal{A}}^{\times 2}\right)$$

by dualizing product $*$:

$$\Delta f(M,N) = \sum_{P} \frac{a_M a_N}{a_P} \left| \left\{ L \subseteq P : \begin{array}{l} L \simeq M \\ P/L \simeq N \end{array} \right\} \right| f(P)$$

<u>Theorem</u> (Green) If the global dimension $\mathrm{gldim}(\mathcal{A}) \leq 1$ $\quad(\mathrm{Ext}^i = 0 \ \forall i \geq 2)$ then $(\mathcal{H}_{\mathcal{A}}, *, \Delta)$ is a (topological) bialgebra. [in fact, if & only if]

[incompatibility of $*$ & Δ is given by an Ext^2]

Remark: counit $\eta f = f(0)$

<u>Example</u> Q quiver := oriented graph (I, E) vert. edges

$$A = \mathrm{Rep}_k(Q) = \left\{ (V_i)_{i \in I},\ (x_e)_{e \in E} : V_i \in \mathrm{Vec}_k,\ x_e \in \mathrm{Hom}(V_i, V_j) \text{ for } i \xrightarrow{e} j \right\}$$

e.g. $Q = \bullet \Rightarrow \mathrm{Rep}\, Q = \mathrm{Vec}_k$

$Q = \circlearrowleft\!\bullet \Rightarrow \mathrm{Rep}\, Q = \left\{ (V, x) \begin{array}{l} \text{vector space} \\ + \text{endomorphism} \end{array} \right\}$

$Q = \circ \longrightarrow \circ \Rightarrow \mathrm{Rep}\, Q = \{ (V_1 \xrightarrow{x} V_2) \}$

Assume Q is a finite Dynkin quiver.

$\rightsquigarrow$ simple Lie algebra $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$

& Cartan matrix $a_{ij} = \dfrac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$

Quantum group (Drinfeld-Jimbo) $v \in \mathbb{C}^\times$

$U_v \mathfrak{g} \supset U_v \mathfrak{n}_+$:

$U\mathfrak{n}_+ = \langle e_i \rangle_{i \in I} \Big/ \begin{array}{l} \text{Serre relations} \\ \mathrm{ad}^{1-a_{ij}}(e_i) = e_j \end{array}$

$U_v \mathfrak{n}_+ = \langle e_i \rangle_{i \in I} \Big/$ q-Serre relations

$$\sum_{l=0}^{1-a_{ij}} (-1)^l \begin{bmatrix} 1-a_{ij} \\ l \end{bmatrix}_v e_i^l e_j e_i^{1-a_{ij}-l}$$

Theorem (Ringel, Green) If Q is a finite Dynkin quiver $\Rightarrow \mathcal{H}_{\mathrm{Rep}_{\mathbb{F}_q} Q} \simeq U_v \mathfrak{n}_+$

$(v^2 = q)$ as bialgebras

$q = 1$? ... above algebras depended polynomially in q, could ask what happens if we set $q = 1$.

Function theory 101

$$f * g(M) = \sum_{N \subseteq M} f(N)\, g(M/N)$$

$$= p_{3*}(p_1^* f \cdot p_2^* g)(M)$$

Here

$$\begin{tikzcd} \mathcal{X}^{(2)} = \{ 0 \subseteq N \subseteq M \} \arrow[r, "p_3"] \arrow[d, "p_1"'] \arrow[dr, "p_2"] & \mathcal{X} \ni M \\ \mathcal{X} \ni N & \mathcal{X} \ni M/N \end{tikzcd}$$

ie $*|_{\mathbb{F}_q}$ has to do with counting points over $\mathbb{F}_q$ in the fiber of p_3, weighted by f, g.

But when it makes sense,

$$|X(\mathbb{F}_q)|\big|_{q=1} = \chi(X(\mathbb{C}))$$

Euler characteristic

So let $\mathcal{A} = \operatorname{Rep} R$, R f.d. algebra over $\mathbb{C}$

$\mathfrak{X}_{\mathcal{A}}$ = moduli stack of objects in $\mathcal{A}$

$$= \coprod_{d \geq 0} \mathrm{Rep}_d(R) / GL_d(\mathbb{C})$$

d-dim reps of R

What should we take as functions on $\mathfrak{X}_d$?

<u>Def</u> A <u>constructible function</u> f on a variety X is a fn. of the form

$$\sum_{i=1}^{r} a_i 1_{Y_i} \quad , \qquad Y_i \subseteq X \text{ locally closed.}$$

$$\int f := \sum a_i \chi(Y_i)$$ definition of integral for constructible functions

$p: X \longrightarrow Y$, $f \in CFun(X) \Rightarrow p_* f \in CFun(Y)$

$$p_* f(y) := \int_{p^{-1}(y)} f = \int_X f \, 1_{p^{-1}(y)} .$$

So define $\mathcal{H}_{\mathcal{A}}$:= constructible functions on $\mathfrak{X}_{\mathcal{A}}$:=

$$\bigoplus_d GL_d\text{-invariant } C\text{-fns on } \mathrm{Rep}_d(R).$$

$$f * g = p_{3*}(p_1^* f \cdot p_2^* g)$$

Theorem 1 This is an associative algebra with unit 1_0.
[Riedtmann / Schofield]

Coproduct: $\Delta : \mathcal{H}_A \longrightarrow CFun(\mathcal{X}_A^{\times 2})$

$$\Delta f(M, N) = f(M \oplus N)$$

... all terms coming from nontrivial extensions vanish at $q=1$ (have free k^* actions!)

Note $\Delta \neq *^t$: not transpose of $*$.

At $q \neq 1$ quantum groups are "self dual / hermaphroditic" - but not at $q=1$

Theorem 2 (Riedtmann, Schofield, Joyce)
$(\mathcal{H}_A, *, \Delta)$ is a topological bialgebra
[irrespective of global dim of A]

Theorem 3 Q Dynkin quiver of finite type
$\Rightarrow \mathcal{H}_{Rep_{\mathbb{C}} Q} \cong U\mathfrak{n}_+$ as bialgebras