Noncommutative projective geometry.

J. T. Stafford,

University of Michigan.

Noncommutative Projective Geometry

J. T. Stafford

University of Michigan.

In recent years a surprising number of significant insights and results in noncommutative algebra have been obtained by using the global techniques of projective algebraic geometry. This talk will survey some of these results.

Thus, we will be interested in using geometric techniques to study graded noncommutative rings.

Definitions:
k = fixed algebraically closed base field. *A connected graded (cg) k-algebra* is a ring $R = k \oplus R_1 \oplus R_2 \oplus \cdots$ such that:

(i) $\dim_k R_i < \infty \ \forall\ i \geq 0$.

(ii) R is generated by R_1 as a k-algebra.

gr-R = category of all fin gen, graded right R-modules $M = \bigoplus_{i \in \mathbb{Z}} M_i$.
qgr-R = gr-R/{fin dim R-modules}.

Intuition: qgr-R = "coherent sheaves" on the imaginary space Proj(R).

Commutative Theory. Fix throughout:

A projective variety (or scheme) X with an invertible sheaf $\mathcal{L}$ and set coh(X) for the category of coherent sheaves on X.

The *homogeneous coordinate ring* is

$$B(X, \mathcal{L}) = k \oplus \bigoplus_{n \geq 1} \mathrm{H}^0(X,\, \mathcal{L}^{\otimes n})$$

Example: $X = \mathbb{P}^1$, with $k(\mathbb{P}^1) = k(u)$. Set $\mathbb{A} = \mathbb{P}^1 \smallsetminus \{\infty\}$ and $\mathbb{A}' = \mathbb{P}^1 \smallsetminus \{0\}$; thus $\mathcal{O}(\mathbb{A}) = k[u]$ and $\mathcal{O}(\mathbb{A}') = k[u^{-1}]$.

Let $\mathcal{L} = \mathcal{O}(1)$ to be the sheaf generated by $x = 1$ and $y = u$; thus $\mathcal{L}(\mathbb{A}) = k[u]$ but $\mathcal{L}(\mathbb{A}') = uk[u^{-1}]$. Then $\mathcal{L}$ has *global sections* $\mathrm{H}^0(\mathbb{P}^1, \mathcal{L}) = k[u] \cap uk[u^{-1}] = k + ku = kx + ky$.

Then $\mathcal{L}^{\otimes 2} = \mathcal{O}(2)$ is defined by $\mathcal{L}^{\otimes 2}(\mathbb{A}) = k[u]$ but $\mathcal{L}^{\otimes 2}(\mathbb{A}') = u^2 k[u^{-1}]$

$\mathrm{H}^0(\mathbb{P}^1, \mathcal{L}^{\otimes 2})$ has basis $1 = x^2,\ u = xy,\ u^2 = y^2$.

Similarly $\mathrm{H}^0(\mathbb{P}^1, \mathcal{L}^{\otimes n})$ has basis $x^n, \ldots, y^n$ and so $B(\mathbb{P}^1, \mathcal{O}(1)) = k[x, y]$.

Think of $\mathbb{P}^1 \rightsquigarrow B(\mathbb{P}^1, \mathcal{O}(1))$ as the converse to $k[x, y] \rightsquigarrow \mathrm{Proj}(k[x, y]) = \{(\lambda_0 x + \lambda_1 y)\} = \mathbb{P}^1$.

Serre's Theorem. *Assume $\mathcal{L}$ is ample;* ie, $\forall$ $\mathcal{F} \in \mathrm{coh}(X)$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by $\mathrm{H}^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$ for $n \gg 0$. *Then:*

(1) $B = B(X, \mathcal{L})$ is a fg noetherian k-algebra.

(2) qgr-$B \sim \mathrm{coh}(X)$ *and* $X = \mathrm{Proj}(B)$.

Noncommutative analogues.

Definitions: Let $\sigma \in \mathrm{Aut}(X)$. Set

$\mathcal{L}^\sigma = \sigma^* \mathcal{L}$ and $\mathcal{L}_n = \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}}$.

The *twisted homogeneous coordinate ring* is

$B = B(X, \mathcal{L}, \sigma) = k \oplus \bigoplus_{i=1}^\infty \mathrm{H}^0(X, \mathcal{L}_n)$.

If $B_n = \mathrm{H}^0(X, \mathcal{L}_n)$ then B has a natural multiplication:

$$\begin{aligned} B_n \otimes B_m \quad &\xrightarrow{\sim} \mathrm{H}^0(X, \mathcal{L}_n) \otimes \mathrm{H}^0(X, \mathcal{L}_m^{\sigma^n}) \\ &\xrightarrow{\phi} \mathrm{H}^0(X, \mathcal{L}_{n+m}) = B_{n+m} \end{aligned}$$

For ϕ, take global sections $\mathrm{H}^0(X, \_\_)$ of

$$\mathcal{L}_n \otimes \mathcal{L}_m^{\sigma^n} \xrightarrow{\sim} \mathcal{L}_{n+m}.$$

Example (cont): $X = \mathbb{P}^1$, $\mathcal{L} = \mathcal{O}(1)$ and $\sigma(\lambda_0 : \lambda_1) = (\lambda_0 : q\lambda_1)$, for $q \in k^*$. Then:

$B(X, \mathcal{O}(1), \sigma) \cong k\{x, y\}/(xy - qyx)$
is the "quantum affine plane" (or "quantum" $\mathbb{P}^1$).

Proof: In $k(\mathbb{P}^1) = k(u)$, $\sigma(u) = qu$. Recall $\mathcal{L} = \mathcal{O}(1)$ is generated by $x = 1$ and $y = u$. Thus $\mathcal{L}^\sigma = \mathcal{L}$ and so $\mathcal{L}_n = \mathcal{L}^{\otimes n} = \mathcal{O}(n)$.

Thus, as vector spaces:

$B(X, \mathcal{O}(1), \sigma) \cong B(X, \mathcal{O}(1), 1) \cong k[x, y]$.

But
$xy = \mathrm{Im}\Big(x \otimes y \to x \otimes y^\sigma = 1 \otimes (qu)\Big) = qu$
yet
$yx = \mathrm{Im}\Big(y \otimes x \to y \otimes x^\sigma = u \otimes 1\Big) = u.$

So $xy = qyx$.

There is an analogue of Serre's Theorem.

Theorem (Artin-Van den Bergh).

Assume $\mathcal{L}$ is σ-ample; ie,

$\forall\ \mathcal{F} \in \mathrm{coh}(X)$, $\mathcal{F} \otimes \mathcal{L}_n$ is generated by its sections for $n \gg 0$. *Then:*

(1) $B = B(X, \mathcal{L}, \sigma)$ is a finitely generated Noetherian k-algebra.

(2) qgr-$B \sim \mathrm{coh}(X)$.

Remark: If X is an irred curve or if $X = \mathbb{P}^n$ then

$$\sigma - ample \ \Leftrightarrow \ ample.$$

For more general X it is a more subtle, but still understood concept (**Keeler**).

Appl. 1; Noncommutative Curves:

Theorem (Artin-S) *Let R be a cg domain such that* $\dim_k R_n$ *grows linearly. Then:*

(1) $R = B(X, \mathcal{L}, \sigma)$, *for an irred curve* X, $\sigma \in \mathrm{Aut}(X)$ *and invertible sheaf* $\mathcal{L}$,
up to a fin dim vector space.

(2) R is noetherian with qgr-$R \sim \mathrm{coh}(X)$.

Thus "NC curves are commutative."

Converse: (Reiten-Van den Bergh) If $\mathcal{C}$ is a Grothendieck category satisfying the basic properties of $coh(X)$ for a smooth irred curve X, then $\mathcal{C} \sim$ qgr-R for some cg ring R such that $\dim_k R_n$ grows linearly.

Consequences: (1) Up to a change of variable the only auts of $k(\mathbb{P}^1)$ are $u \mapsto qu$ and $u \mapsto u+1$. Thus the only quantum $\mathbb{P}^1$'s are $xy = qyx$ and $xy = yx + y^2$.

(2) If $|\sigma| < \infty$ then R is a finite module over its centre.

If $|\sigma| = \infty$ then X is rational or elliptic and R has at most 2 proper homogeneous prime ideals.

Remarks: Set

$$\text{GKdim}(R) = \liminf\{\alpha : \dim(\sum_{i=0}^{n} R_i) \leq n^\alpha\}$$

The last theorem works for $GKdim(R) = 2$. If R is a cg domain with $GKdim(R) = 1$, then $R = k[x]$ (up to a fin dim vector space).

There are no cg domains R with $1 < Gkdim(R) < 2$ (Bergman) or $2 < Gkdim(R) < 3$ (Smoktunovicz), so the next case is:

Appl. 2 : Noncommutative $\mathbb{P}^2$'s.

In the 80's **Artin & Schelter** were trying to classify noncom analogues of $k[x, y, z]$ (Artin-Schelter regular rings). There was one example they could not analyse.

Example: (**The Sklyanin Algebra**) Let $(a : b : c) \in \mathbb{P}^2 \smallsetminus K$ for a (known) set K. Then $S = S(a, b, c) =$

$$\frac{k\{x_0, x_1, x_2\}}{\big(ax_ix_{i+1} - bx_{i+1}x_i - cx_{i+2}^2 : i \in \mathbb{Z}/(3)\big)}.$$

Artin, Tate and Van den Bergh where eventually able to understand S "geometrically" by showing that S mapped onto some $B(E, \mathcal{L}, \sigma)$ (where E is an elliptic curve). This then controlled the structure of S and enabled them to understand its properties.

So why do twisted homogeneous coordinate rings keep turning up?

Definition: A *point module* is a cyclic graded R-module $M = M_0 \oplus M_1 \oplus \cdots$ with $\dim M_i = 1 \ \forall\ i \geq 0$.

If R is a commutative cg domain, then:

$$\{\text{Point modules for } R\} \ = \ \{\text{graded factor rings } R/P \cong k[x] \text{ of } R\} \leftrightarrow \text{points in } X = \operatorname{Proj}(R).$$

Point Modules for $S = S(a, b, c)$.

If M is a point module, write $M_i = m_i k$ and $m_i x_j = \lambda_{ij} m_{i+1}$ for some $\lambda_{ij} \in k$.

If $f = f(x_i x_j) \in S_1 \times S_1$ is a quadratic reln for S, then $0 = m_0 f = f(\lambda_{0i}\lambda_{1j})m_2$.

So $f(\lambda_{0i}\lambda_{1j}) = 0$. This defines $\Lambda \subseteq \mathbb{P}(S_1^*) \times \mathbb{P}(S_1^*)$ and Λ parametrizes *truncated point modules of length* 3:
$M = M_0 \oplus M_1 \oplus M_2$
(with M cyclic and $\dim_k M_i = 1$).

Steps of ATV's Proof:

(1) Λ is the graph of an aut σ of an elliptic curve $E \overset{\iota}{\subseteq} \mathbb{P}^2 = \mathbb{P}(S_1^*)$.

(2) E param all point modules over S.

(3) σ is the *shift functor* on point modules: $M \mapsto M[1]_{\geq 0} = M_1 \oplus M_2 \oplus \cdots$

(4) By the construction there is a ring hom $\pi : S \to B = B(E, \mathcal{L}, \sigma)$.

Theorem (Artin-Tate-Van den Bergh)

(i) π is surjective and $B \cong S/gS$ $(g \in S_3)$.

(ii) S is AS-regular of dim 3*; that is:*

(a) $\operatorname{gldim} S = 3$.

(b) S has polynomially bounded growth.

(c) $\operatorname{Ext}^i(k, S) = \begin{cases} 0, & \text{if } i \neq 3; \\ k, & \text{if } i = 3. \end{cases}$

Theorem (ATV), cont.

(iii) Conversely, if R is AS regular of dim 3 then R is a noetherian domain and either:

(a) $R = B(X, \mathcal{L}, \sigma)$ *with* $X = \mathbb{P}^2$

(b) $R = B(X, \mathcal{L}, \sigma)$ *with* $X = \mathbb{P}^1 \times \mathbb{P}^1$ *or*

(c) $R \twoheadrightarrow B(E, \mathcal{L}, \sigma)$ *for E a curve. In this case R is determined by* $\{E, \mathcal{L}, \sigma\}$.

Remarks: (i) If R is AS-regular of dimension 3 and Hilbert series $1/(1-t)^3$ we call R—or qgr-R—a *noncommutative* $\mathbb{P}^2$.

(ii) This holds for $S = S(a, b, c)$. Think of qgr-$S(a, b, c)$ as a NC $\mathbb{P}^2$ with an embedded elliptic curve E.

(iii) (**Bondal and Polishchuk**) These are the "only " NC $\mathbb{P}^2$'s.

Program to classify all NC surfaces ($\equiv$ cg domains R with $GKdim R = 3$.):

1) *Classify up to birational equivalence* ($\equiv$ classify their graded division rings).

Conj (Artin): These are known.

2) *Classify the minimal models.*

In the rational case these are $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^1$ bundles over $\mathbb{P}^1$. Their NC analogues have been classified by **Bondal-Polishchuk** & **Van den Bergh**.

3) *Prove that all NC surfaces can be obtained from minimal models via blowing up and down.*

This is wide open, although **Van den Bergh** has an analogue of blowing up and down.

(You will need extra cnds on R, since this is really a program for "smooth surfaces.")

Application 3. Weyl algebras and Calogero-Moser Space. The Weyl algebra $A_1(\mathbb{C}) = \{\sum f_i(x)\delta^i\}$ for $f_i(x) \in \mathbb{C}[x]$, $\delta_i = \frac{d}{dx}$.

Theorem (Cannings-Holland, Berest-Wilson) Isomorphism classes of right ideals of A_1 correspond to Wilson's adelic Grassmannian $GR^{\mathrm{ad}} = \coprod_{n>0} \mathcal{C}_n$, where $\mathcal{C}_n$ is a *completed Calogero Moser space*: $\mathcal{C}_n = \{X, Y \in M_n(\mathbb{C}) : rk([X,Y] - I_n) \leq 1\}$.

Projective approach: (after **Le Bruyn, BW, Baranovsky-Ginzburg-Kuznetsov..**) Form the *homogenised Weyl algebra* H, generated by x, y, z where z is central and $yx - xy = z^2$. This is a NC $\mathbb{P}^2$, with factor $B = H/(z) = B(\mathbb{P}, \mathcal{O}(1), Id)$. The problem becomes to classify reflexive right ideals I of H with $I/zI \cong B$, or, equivalently, their images in qgr-H: the "locally free sheaves" $\mathcal{L}$ that are trivial at infinity: $\mathcal{L}/z\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^1}$.

A torsion-free (tf), rank 1 module $\mathcal{L} \in \text{qgr-}H$ (that is, the image of a right ideal of H) has invariants: the first Chern class c_1 (a unique shift $\mathcal{L}[n]$ has $c_1 = 0$) and the Euler char $\chi(\mathcal{L}) = \sum(-1)^j \dim_k \operatorname{Ext}^j_{qgr-H}(H, \mathcal{L})$.

Theorem: (1) For all $n \geq 0$ there is a fine moduli space $\mathcal{M} = \mathcal{M}^{ss}_H(1, 0, 1-n)$ for equivalence classes of rank one tf modules $\mathcal{L}$ in qgr-H with $c_1 = 0$ and $\chi = 1 - n$.

(2) $\mathcal{M}$ is a (smooth irreducible projective) deformation of the Hilbert scheme $(\mathbb{P}^2)^{[n]}$ of n points in $\mathbb{P}^2$.

(3) $\mathcal{M} \supset \mathcal{C}_n$, which is a deformation of $(\mathbb{A}^2)^{[n]}$, and parametrizes equivalence classes of locally free sheaves in qgr-H with $\chi = 1 - n$ and trivial at infinity.

Analogues hold for all NC $\mathbb{P}^2$s (**Nevins-S** and **De Naeghel-Van den Bergh**).