Paul Hacking - Moduli of Complex Surfaces

Note Title 3/24/2009

Classification of alg. varieties / $\mathbb{C}$ (complex manifolds) → discrete part: topological invariants

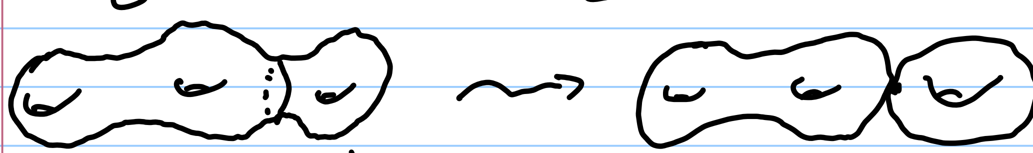
→ continuous: moduli spaces of varieties w/ fixed discrete invariants

Example Curves / $\mathbb{C}$ (Riemann Surfaces)

1. Discrete: $\exists!$ topological invariant = genus = # of handles

2. Moduli space of curves of genus $g \geq 2$ $\mathcal{M}_g$, complex orbifold [locally $\mathbb{C}^n$/finite group] of dimension $3g-3$.

NOT compact: smooth curves can degenerate to singular ones



vanishing cycle = S^1 crushed to a point.

Deligne-Mumford '69:

$\exists$ compactification $\mathcal{M}_g \subset \overline{\mathcal{M}}_g$ s.t. points at boundary correspond to singular curves:

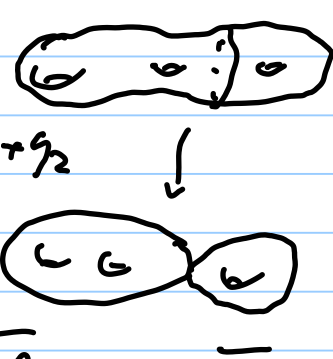
1. Singularities are nodes (locally $(xy=0) \subset \mathbb{C}^2$)
2. "stability": $|\mathrm{Aut}(C)| < \infty$.

explicitly if $\mathbb{P}^1 \subset C$ have at least 3 nodes ...

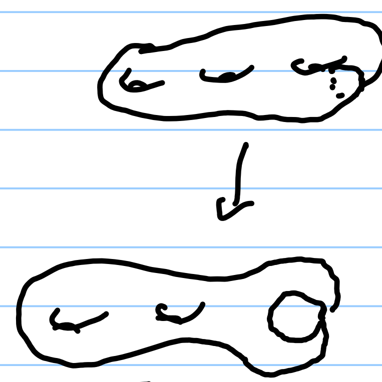
Also $\overline{\mathcal{M}}_g$ is a complex orbifold, & boundary $B = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ is a normal crossing divisor (locally $\prod x_i = 0$)

Finally, components of B are indexed by topological types of degenerations.

eg codimension 1:



$g = g_1 + g_2$



$\overline{\mathcal{M}}_{g_1,1} \times \overline{\mathcal{M}}_{g_2,1}$ $\qquad$ $\overline{\mathcal{M}}_{g-1,2}$

Q: What about complex surfaces?

1. Discrete: everything known except for surfaces of general type:

X surface, ω_X canonical line bundle
- sections are holomorphic 2-forms

$\Rightarrow$ map $X \overset{\varphi_N}{\dashrightarrow} \mathbb{P}^M$ given by all sections $\Gamma(X, \omega_X^{\otimes N})$ - X is general type if φ_N is birational for $N >> 0$.

If $\bar{X} = \mathrm{Im}(\varphi_N) \subset \mathbb{P}^M$ then $\omega_{\bar{X}}$ is ample & $\bar{X}$ has ADE singularities (mild)

Key invariants: Chern numbers $c_1^2 = K_X^2 > 0$,
$c_2 = e(X)$ Euler number (& π_1)

NOT known which can occur.
General results - eg Yau inequality $c_1^2 \leq 3c_2$

M. Nöther inequality $5c_1^2 \geq c_2 - 36$

congruence $c_1^2 + c_2 \equiv 0 \ (12)$.

"geography of surfaces"

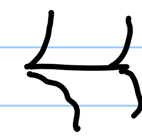
Can have arbitrarily many connected components for fixed c_1^2 & c_2!

2. Continuous part Fix c_1^2, c_2

$\exists$ moduli space $\mathcal{M}$ of surfaces of general type w/ given invariants

WARNING: $\mathcal{M}$ may be highly singular, & may be disconnected (but finitely many components)

$\exists$ compactification $\mathcal{M} \subset \overline{\mathcal{M}}$
(Kollár, Shepherd-Barron) analogous to $\mathcal{M}_g \subset \overline{\mathcal{M}}_g$ with $B = \overline{\mathcal{M}} \setminus \mathcal{M}$ parametrizing singular surfaces.

1. Singularities: "semilog canonical"
includes $(xy=0) \subset \mathbb{C}^3$

& orbifold singularities
& some others (isolated)

2. Stability: ω_X is ample
(dualizing sheaf .... a $\mathbb{Q}$-line bundle)

Kollár: $\overline{\mathcal{M}}$ is projective

Line bundle over $\overline{\mathcal{M}}$:

$$\begin{array}{cl} \mathcal{U} & \text{universal} \\ \downarrow p & \text{family} \\ \mathcal{M} & \end{array}$$

$L = \det p_* \omega_{\mathcal{U}/\mathcal{M}}^{\otimes N}$ $\quad N \gg 0$

- fibres are $\Lambda^{top}\, \Gamma(\omega_{\mathcal{U}}^{\otimes N})$

Why is this ample? case $N=1$

$\Gamma(\omega) = H^{2,0} \subset H^2(X, \mathbb{C})$

Griffiths: these have correct curvature properties so get a nonnegative bundle.

Q Why is M singular?

Kodaira-Spencer theory:

X complex manifold. Cover X by complex balls

$X = \bigcup U_i$

$H^1(T_X) = \check{H}^1(\{U_i\}, T_X)$ Čech cohomology

= tangent space to deformations of X

$\{g_{ij}\} \in H^1(T_X)$: g_{ij} is a section of tangent bundle on overlaps of U_i & U_j

$\Rightarrow$ telling us how to change gluing

Then quotient out by reparametrization of the U_i.

$H^2(T_X)$ contains obstructions to lifting deformations to higher order:

Given a first order deformation, lift g_{ij} arbitrarily to $\tilde{g}_{ij}$, gluing to order 2 $\longrightarrow$ cocycle condition no longer satisfied:

$g_{ik} = g_{jk} \circ g_{ij}$ breaks down to next order.

Can fix this as long as class in $H^2(T_X)$ vanishes:

$$0 = (\tilde{g}_{ik}^{-1} \circ \tilde{g}_{jk} \circ \tilde{g}_{ij}) \in H^2(T_X)$$

$\rightsquigarrow$ get deformation space

$$\mathrm{Def}\, X \subset H^1(T_X) \simeq \mathbb{C}^n$$

given by at most $\dim H^2(T_X)$ equations

In particular, $\mathrm{Def}\, X$ is smooth if $H^2(T_X) = 0$.

For curves, $H^2(T_X)$ always vanishes... but for surfaces typically it's nonzero.

General type: $H^2(T_X) = H^0(\Omega_X \otimes \underbrace{\omega_X}_{\text{ample}})$

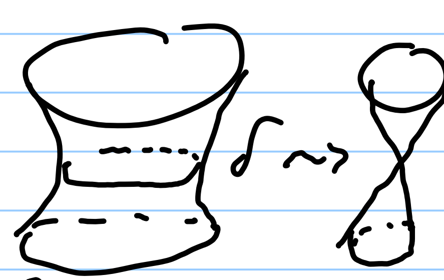
– very often nonzero!

Q What are the boundary components of $\overline{M}$?

(Assume we have a reasonably nice component of M)

A: Two main cases:

1. Lefschetz degeneration



$\delta = S^2$ vanishing cycle crushed to a singularity $x^2+y^2+z^2=0$

$(\delta \in H^2(Y,\mathbb{Z}) \quad \delta^2 = -2)$

2. Wahl degeneration (also happens in codim one but not a complete intersection)

local model $\quad$ $X \subset \mathcal{X} = (xy = z^n + t)$

$$\begin{array}{ccc} X & \subset & \mathcal{X} \\ \downarrow & & \downarrow \\ 0 & \in & \Delta \end{array} \qquad \subset \mathbb{C}^3_{x,y,z} \Big/ \mathbb{Z}/n\mathbb{Z} \times \Delta_t$$

$\zeta \in \mathbb{Z}/n\mathbb{Z}$ action:

$$(x, y, z) \longmapsto (\zeta x, \zeta^{-1} y, \zeta^a z)$$

$$(a, n) = 1$$

$$X = (xy = z^n) \Big/ \mathbb{Z}/n = \mathbb{C}^2_{u,v} \Big/ \mathbb{Z}/n^2\mathbb{Z}$$

$$x = u^n \quad y = v^n \quad z = uv.$$

A_{n-1} singularity: has many deformations

$$xy = z^n + a_{n-2} z^{n-2} + \cdots + a_0$$

but to preserve group action can only have $xy = z^n + a_0$

one parameter $\rightsquigarrow$ codim one.

<u>Problem</u> Enumerate Wahl degenerations of a given surface.

[talk on Thursday: related to vector bundles on smooth fiber].

Application: Lee + Park 2007:
"new" example of surface of general type via Wahl degeneration:

$c_1^2 = 2 \quad b_2^+ = 1 \quad \pi_1 = 0$

Idea: 1) wrote down rational surface X w/ Wahl singularities s.t.
K_X is <u>ample</u> (uses elliptic fibrations)
... motivations from Seiberg-Witten theory

2. Show $H^2(T_X) = 0 \Rightarrow$
$\exists$ smoothing of X ... can globalize any local deformation of the singularities

3. Compute topology of smoothing using Milnor fibers.

<u>Examples</u>: 1. Very nice theory for K3s:
$c_1 = 0 = K_X \quad , \quad \pi_1 = 0$

2. $b_2^+ = 1$, $\pi_1 = 0$, general type:

Freedman '82: homeomorphic to del Pezzo surfaces $= Bl^n \mathbb{P}^2$ $n \leq 8$ or $\mathbb{P}^1 \times \mathbb{P}^1$, but not diffeomorphic.

(Kodaira dimension is diffeo. invariant!)

examples
- Barlow surface '85 $c_1^2 = 1$
- Lee + Park '07 $c_1^2 = 2$.

Q: What is $D(X)$? (derived category)

expected dim of moduli space

$$h^1(T_X) - h^2(T_X) = -\chi(T_X) = 2n - 8$$

3. Surfaces on the Nöther line (c_1^2 minimal for given c_2):

Horikawa ('75) gives complete classification

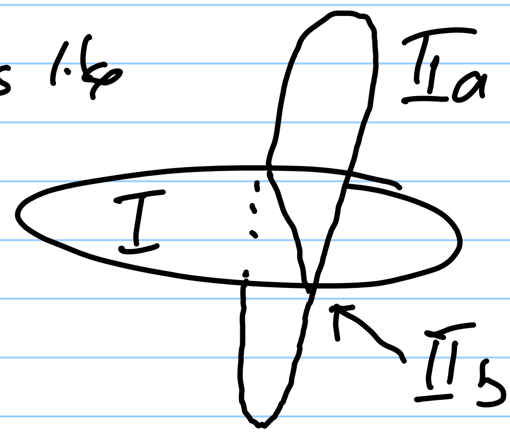
Ex. $c_1^2 = 5$, $c_2 = 55$, $b_1 = 0$
(eg quintic surface in $\mathbb{P}^3$)

Classification: such are either

I. quintic surface $X \subset \mathbb{P}^3$
given by ω_X

II. blowup $\tilde{X} \xrightarrow{2:1} Q \subset \mathbb{P}^3$ quadric
$\downarrow$
X with a. Q smooth
b. Q singular

Moduli looks like



$(xy=0) \subset \mathbb{C}^{41}$
$\shortparallel$
$H^1(T_X)$