

# Nigel Higson - K-homology & Index Theory

Note Title

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## I: K-theory from the viewpoint of functional analysis

M closed manifold

D first order PDO of Dirac type:

$$\text{locally } D = \sum a_j \frac{\partial}{\partial x_j} (+ b)$$

where  $a_j$  matrices work

$$a_i a_j^* + a_j a_i^* = \begin{cases} 2I & i=j \\ 0 & \text{else} \end{cases}$$

e.g.  $D = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y}$  (Cauchy-Riemann)

Fundamental fact:  $\dim \ker D < \infty \quad \left. \right\} \text{Fredholm}$   
 $\& \dim \text{coker } D < \infty \quad \left. \right\} \text{Fredholm}$

$$\begin{aligned} \text{Index}(D) &= \dim \ker D - \dim \text{coker } D \\ &= \text{Same for } C^\infty, L^2, C^{-\infty}, \dots \end{aligned}$$

Also  $\text{coker } D \cong \ker D^*$  (formal adjoint) etc.

Message In index theory (à la Atiyah-Singer)

it ought to be useful to study Fredholm operators abstractly, via functional analysis  
- in particular via  $L^2$  methods (Hilbert spaces, operators,  $C^*$  algebras, etc.)

Today: functional analysis & K-theory  
(Akhiezer, '60s)

Fredholm operators : basics :

Suppose  $\bar{T} : H_0 \rightarrow H_1$  bounded Fredholm opn  
between Hilbert spaces  
— practically invertible! — — —

$\begin{pmatrix} T & P_{\ker T^*} \\ P_{\ker T} & 0 \end{pmatrix} : H_0 \oplus \ker T^* \rightarrow H_1 \oplus \ker T$   
is invertible

$\Rightarrow \{ \text{all Fredholm operators} \}$  open  
in bounded operators with norm topology.

- Index is locally constant
- Fredholm + compact is Fredholm with same index

— summarize as Atkinson's Theorem:

Fredholm  $\hookrightarrow$  invertible modulo compacts.

$\text{Fred} = \text{Fred}(H_0, H_1) = \text{all Fredholms}$   
- reasonable, big topological space.

Assume  $H_0, H_1$   $\infty$ -dim (separable)

Theorem (Atiyah - Janich)  $X$  compact  $\Rightarrow$

$K(X) \cong [X, \text{Fred}] =$  homotopy classes  
of maps  $X \rightarrow \text{Fred}$   
 $K$ -theory of topological vector bundles

Idea of proof Suppose we have

$F: X \rightarrow \text{Fred}$  family of Fredholm-s

& suppose  $\dim \ker F_x$  is a locally constant  $\mathbb{Z}$ -valued function on  $X$

Then  $\{\ker F_x \subset H_0\}$  constitute a vector bundle on  $X$ , as do  $\ker F_x^* \subset H_1$ ,  
 $\Rightarrow$  can define  $\text{Index}(F) =$   
 $[\ker F] - [\ker F^*]$

What if  $F_x$  is invertible?

set 0 index ... Kuiper: in this case  
 $F$  is homotopic to a constant

[don't know any explicit construction!!?]

What if  $\dim(\ker F_x)$  not locally constant?

- then replace  $F$  by  $FP$  where  
 $P: H_0 \rightarrow H$ , is a finite codim projection  
 to fix the problem. (can make the  
 family of kernels constant this way!)

... &  $FP$  is homotopic to  $F$  by  
 straight-line interpolation.

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### Pseudolocality (Atiyah)

$D$  as before  $D: C^\infty(M, S_0) \rightarrow C^\infty(M, S)$   
 $S_i$ : smooth vector bundles

If isn't defined as an operator  
 $H_0 = L^2(M, S_0) \rightarrow L^2(M, S_1) = H$ ,

The f.x:  $D$  can be "extended" (domain  
 enlarged) so that

$$u_n \rightarrow u \quad u_n \in C^\infty, \quad u \in L^2, \text{ converges in } L^2 \\ \& D u_n \rightarrow v \quad v \in L^2$$

$$\Rightarrow u \in \text{domain} \quad \& \quad Du = v.$$

" $D$  is a closed operator".

&  $\exists$  minimal such extension.

(domain is a Sobolev space)

Now (von Neumann)  $D$  has a "polar decomposition"

$$D = A F \quad (\text{think } z = r e^{i\theta})$$

$A = \sqrt{DD^*}$ ,  $F$  is a partial isometry

(from  $\text{Ker}^\perp \rightarrow \text{Range set isometry}$ )

$$\text{Ker } F = \text{ker } D, \quad \text{Ker } F^* = \text{ker } D^*.$$

$F$  is bounded & Fredholm, same index as  $D$ .

(induce trace homology  $A$  to  $\text{Id}$   
via red semicircle operators)

Theorem I If  $f$  is a continuous function  
on  $M$  then the commutator

$[F, f]$  is compact

... so  $F$  is "practically" linear / functions,

Why the fuss?

A vector bundle  $E$  on  $M$  can be realized  
as a projection valued map

$$P: M \longrightarrow M_n(\mathbb{C}) \quad P(m)^2 = P(m) \quad V_m$$

$$\text{s.t. } E_m = P(m) \mathbb{C}^n.$$

We can consider  $P$  as a projection in  
 $M_n(C(\mathbb{M}))$  matrices of cont. functions  
& so as a projector operator

$$P_0 : H_0 \oplus H_0 \oplus \dots \oplus H_0 \longrightarrow H_0 \oplus \dots \oplus H_0$$

$$P_1 : H_1 \oplus \dots \oplus H_1 \longrightarrow H_1 \oplus \dots \oplus H_1$$

$$\& P_1(F \dots F)P_0 : \text{range}(P_0) \rightarrow \text{range}(P_1)$$

is Fredholm

$$\implies \text{Index}_D : K(M) \rightarrow \mathbb{Z} = K(pt)$$

& more generally (via Atiyah-Jänich)

$$\text{Index}_D : K(M \times X) \rightarrow K(X)$$

functorial in  $X$

The maps above determine a class  $[D]$  in  
 $K$ -homology  $K_0(M)$  (homology theory  
associated abstractly to cohomology theory  $K$ )

Really only need pseudo-local bounded  
Fredholm  $F$  as above

In fact every class in  $K_0(M)$  ranges  
 this may (more or less)  
 ... ie can define cycles to be elliptic operators!

Question how to define K-homology from  
 first principles via these Fredholm  
 $C(M)$ -modules?

A Fredholm  $(M)$ -module [cons] is

$$F: \mathbb{H}_b \rightarrow \mathbb{H}, \quad \text{s.t. } [F, F^*] \text{ compact}$$

$\circlearrowleft \quad \circlearrowright$   
 $C(M)$

The answer (Kasparov) :

Def A continuous field of Hilbert spaces

$\{H_x\}_{x \in X}$  ( $X$  loc. cpt Hausdorff), is

- a set of Hilbert spaces  $x \in X$

- a family of sections, called

continuous sections, s.t.

- $x \mapsto \|s(x)\|$  is continuous (s norm)

- cts sections are a  $C(X)$ -module  
 etc (eg closed under uniform limit)

A bounded operator of continuous fields is  
 .... what you think.... (except  
 we require the pointwise adjoint also  
 continuous.)

A compact operator on  $\{f_i\}_{i \in X}$

is a norm limit of operators of the form

$$S \mapsto \sum_{i=1}^n \langle S, f_i \rangle r_i$$

where  $f_i, r_i$  are  $C_0$  sections (norms  
 vanish at  $\infty$ )

A Fredholm operator is an operator invertible  
 mod. compact

[These defns due to Dixmier-Dailey]

Examples  $F: H_0 \rightarrow H_1$ , a single  
 Fredholm operator

Build continuous fields  $H_0, H_1$ , over  $[0,1]$

with fibers  $\begin{cases} \cdot : H_0 & + \neq 0 \\ 0 & + = 0 \end{cases}$

$$\begin{cases} \cdot : H_0 & + \neq 0 \\ 0 & + = 0 \end{cases}$$

& sections: continuous maps  $[0,1] \rightarrow H_0$   
 vanishing at 0 (& same for  $H_1$ ).

If  $F$  is invertible  $\rightsquigarrow$

define  $\widetilde{F} : H_0 \rightarrow |F|$ , so to be  
fiberwise  $F \Rightarrow$  this is Fredholm.

$\rightsquigarrow$  homotopy between  $F$  & 0  
in sense of continuous fields!

... this would be Fredholm if  $F$  not  
invertible!!

Theorem (improved Atiyah-Jänich)

$K^0(X) =$  homotopy classes of Fredholm families

(even for  $X$  just locally compact)

... easier than Atiyah-Jänich:  
don't need Kührer

Theorem (Kasparov)

$K_0(X) =$  homotopy classes of  
Fredholm  $C(X)$  modules

[homotopy means cont. field over  $[0, 1]$ ].