

Lecture 4

4.1

The Baum-Connes Conjecture

(These notes are brief; see the survey papers indicated on the web page for more details.)

Actually, this turned out not to be so!

A families index construction

Fix $\Lambda \subseteq V$ a lattice in a (real) vector space

Λ = free abelian group

V/Λ = tors.

Define a map

$$m: K_*^{\text{geom}}(V/\Lambda) \longrightarrow K^*(V^*/\Lambda^*),$$

where

$$\left\{ \begin{array}{l} V^* = \text{dual vector space} \\ \Lambda^* = \text{dual lattice} = \{ v^* \in V^* \mid \langle v^*, \lambda \rangle \subseteq \mathbb{Z} \} \\ \text{so } V^*/\Lambda^* = (\text{dual}) \text{ forms} \end{array} \right.$$

as follows:

Given (Π, E, f) and $v^* \in V^*/\Lambda^*$

(i) Think of v^* as a character of Λ :

$$v^*(\lambda) = e^{2\pi i \langle v^*, \lambda \rangle}$$

So think of $v^* \in V^*/\Lambda^*$ as a flat Hermitian line bundle on V/Λ (a space with fundamental group Λ). Call this line bundle L_{v^*} .

(ii) Use f to pull back L_{v^*} to Π

(iii) Form $\left\{ \text{D}_{\text{rac}}_{\Pi, E \otimes L_{v^*}} \right\}_{v^* \in V^*/\Lambda^*}$

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This is a family of Dirac operators
parametrized by $\sqrt{\epsilon}/\lambda^*$.

(iv) Take the index of this family:

$$\mu(M, E, F) := \text{Index}(\{D_{M, E, \lambda_v}\}_{v \in V^*/\lambda^*})$$

This gives the required map.

Remark The map can also be defined on Atiyah's cycles.

The map has some important properties

(a) It is an isomorphism. More on this soon.

(b) For a cycle (M, E, F) , if M is spin (not just $spin^c$) and E is trivial, and F M has a positive scalar

curvature metric, then

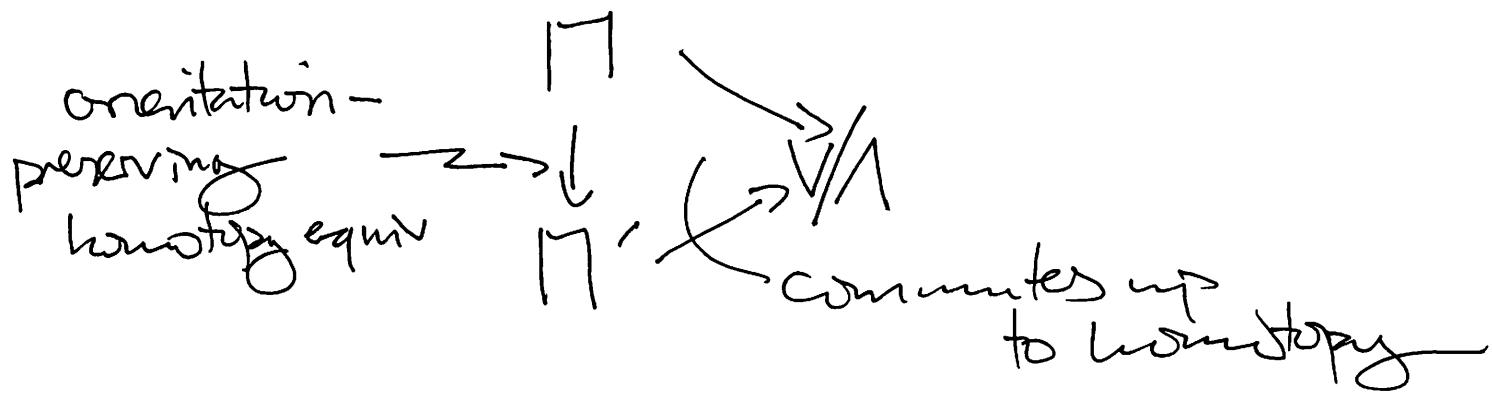
$$\mu(\eta, E, f) = 0$$

(because, by Lichnerowicz's formula, all of the $D_{\text{Dirac}}|_{L^\infty}$ are invertible).

(c) For a cycle (η, S^*, f) , where S = spinor bundle on η (with grading forgotten), the class $\mu(\eta, S^*, f)$ depends only on the oriented homotopy class of

$$\eta \longrightarrow V/\Lambda .$$

So given



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Roughly speaking, this is because
 $D_{S^*} = \text{Signature operator on}$
 Π , so the index depends only
on the cohomology of Π (with
twisted coefficients).

Putting these things together (actually
using only the injectivity of μ) we
get for example:

- There is no positive scalar curvature metric on the torus V/Λ .
- If Π, Π' are oriented-homotopy equivalent, ~~then~~

$$\begin{array}{ccc} \Pi & \xrightarrow{f} & B\Lambda \\ h \downarrow \sim & \nearrow f' & \\ \Pi' & & \end{array} \quad \alpha \in H^*(B\Lambda).$$

then

$$\therefore \int_{\Pi} L(\Pi) \cdot f^*(\alpha) = \int_{\Pi'} L(\Pi') \cdot f'^*(\alpha)$$

Fields of Hilbert spaces versus Hilbert modules

X = compact or weakly compact space

$\Gamma(H)$ = continuous sections of a cts field of Hilbert spaces (vanishing at infinity if X not compact).

This carries
(right)

- (i) a $C_0(X)$ -module structure
- (ii) a $C_0(X)$ -valued inner product
(the pairwise inner product of sections)

One has

$$\langle s, s \rangle \geq 0$$

$$\langle s, s \rangle = 0 \iff s = 0$$

$$\langle s, t \rangle = \langle t, s \rangle^*$$

$$\langle s, tf \rangle = \langle s, t \rangle f$$

etc

plus completeness in the norm

$$\|s\| = \|\langle s, s \rangle\|_{\ell^2}^{1/2}$$

synonym of a function.

There are the axioms for a
Hilbert $C(X)$ -module. We
 can define the concept of
 Hilbert A -module for any C^* -algebra A .

We can recover the fibres of a
 continuous field by

$$H_x \cong \Gamma(H) \otimes_{C_0(X)} \mathbb{C}_x$$

where \mathbb{C}_x is \mathbb{C} with $C_0(X)$ -module
 structure given by eval at $x \in X$.

So continuous fields over X and
 Hilbert $C_0(X)$ -modules are the
 same thing.

Baum-Cornes (-Kasparov) Assembly May

π = discrete group.

$C^*(\pi)$ = group C^* -algebra*

Let $M = C^*(\pi)$, viewed as a right $C^*(\pi)$ -module only (not an algebra). It is a Hilbert $C^*(\pi)$ -module

$$\langle m_1, m_2 \rangle = m_1^* m_2.$$

Each elt of $C^*(\pi)$ gives an endomorphism of M by left multiplication.

In particular each element $g \in \pi$ gives a unitary automorphism of M by left multiplication.

* For experts: the nuance between full and reduced group C^* -algebras will be ignored.

Form

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$$L = E\pi \times \mathcal{M}.$$

$\xrightarrow{\pi}$ π
universal
principal
 π -space

This is a bundle over $B\pi = E\pi/\pi$ with $C^*(\pi)$ -module fibers; it is called the Mischenko line bundle.

It is the universal flat unitary line bundle over $B\pi$. Given any such (with Hilbert module fibers) we obtain a unitary representation of π , hence a $C^*(\pi)$ -representation P from its holonomy and the bundle can be recovered as

$$L \underset{P}{\circledast} H.$$

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By repeating the construction for $\Lambda = \pi$ given above, we get

$$K_*^{\text{geom}}(B\pi) \longrightarrow K_*(C^*\pi)$$

where

$K_*(C^*\pi)$ = C^* -algebra K-theory
of $C^*(\pi)$

\vdash := homotopy classes
of Fredholm operators
on Hilbert $C^*(\pi)$ modules.

There is an Atiyah-Jänich theorem
to relate this to "ordinary" K-theory:

$K_0(A) \cong$ ordinary algebraic
K-group of A .

$$\begin{aligned} K_1(A) &\cong \pi_0(GL_{\infty} A) \\ &= \varinjlim_{n \rightarrow \infty} \pi_0(GL_n A) \end{aligned}$$

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By the way one has natural isomorphisms

$$\pi_j^*(\mathrm{GL}_\infty A) \cong \pi_{j+2}(\mathrm{GL}_\infty A)$$

Bott Periodicity

and in addition

$$K_0(A) \cong \pi_1(\mathrm{GL}_\infty A).$$

Strong Novikov Conjecture:

The map

$$\mu: K_*(B\pi) \longrightarrow K_*(C^*\pi)$$

is rationally injective (injective after tensoring over \mathbb{Z} with \mathbb{Q}).

If π has torsion (e.g. π finite) then $K_*(B\pi)$ is \mathbb{Z} plus lots of torsion, while $K_*(C^*\pi)$ is a free abelian group on the irreducible reps of π . So the SNC is the best one can hope for without modifying

the map μ .

Baum & Connes have worked out how to appropriately modify μ . But for π torsion-free, no modification is needed.

Baum-Connes conjecture for torsion free discrete groups.

If π is torsion-free the assembly map

$$\alpha: K_0(B\pi) \rightarrow K_0(C^*\pi)$$

is an isomorphism.

A Duality Argument

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Return to

$$\pi = \lambda$$

$$B\pi = V/\Lambda$$

Why is the assembly map an isomorphism here?

Well, it can be calculated using the index theorem (for families).

A better perspective: our map is

$$K_*(V/\Lambda) \longrightarrow K^*(V^*/\Lambda^*)$$

But

$$K_*(V/\Lambda) \cong K^*(V/\Lambda)$$

K-hornsway

via Poincaré duality (possibly there is a dimension shift)

We get therefore

$$K^*(V/\Lambda) \longrightarrow K^*(V^*/\Lambda^*)$$

(The map is easy to describe:
 Let D be the Dirac operator on V/Λ . The
 map takes the class of a vector bundle
 E over V/Λ to the index of the family
 $\{D_{E \otimes L_{V^*}}\}_{V^* \in V^*/\Lambda^*}.$)

Having expressed the assembly map μ in this
 way a natural candidate for the inverse
 arises — we can try the very same map
 but for V^*/Λ^* in place of V/Λ (note that
 $V^{**} = V$ and $\Lambda^{**} = \Lambda$).

This is indeed the inverse. We have
 identified the Baum-Connes isomorphism
 with Fourier-Mukai duality (in topological
 K-theory.)

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A closer look suggests a possible generalization of the duality isomorphism to a much broader class of groups—going well beyond free abelian groups.

Clearly the object of central concern in the free abelian case is the family of Dirac-type operators $\{D_{L_v}\}$ on the torus V/Λ , parametrized by the dual torus V^*/Λ^* . These operators act on (smooth) sections of the continuous field

$$\{L^2(V/\Lambda, S \otimes L_v)\}_{v^* \in V^*/\Lambda^*}.$$

The continuous field determines a Hilbert $C(V^*/\Lambda^*)$ -module and hence, thanks to the Fourier isomorphism $C(V^*/\Lambda^*) \cong C^*(\Lambda)$, a Hilbert $C^*(\Lambda)$ -module. Let us compute what it is.

The continuous field is generated by
 the smooth sections of the bundle over
 $(V/\Lambda) \times (V^*/\Lambda^*)$ whose restriction to $(V/\Lambda) \times \{v^*\}$
 is the flat line bundle $S \otimes L_{V^*}$. A
 smooth section of $S \otimes L_{V^*}$ is the same
 thing as a smooth function

(The spinor bundle
 over V/Λ is constant,
 and from here on
 S denotes a single
 fiber vectorspace.)

$$f_{V^*} : V \longrightarrow S$$

such that

$$f_{V^*}(v+k) = e^{2\pi i \langle v^*, k \rangle} f_{V^*}(v)$$

for all $k \in \Lambda$.

The functions f_{V^*} combine to form a
 single function

$$f(v) = \int_{V^*/\Lambda^*} f_{V^*}(v) \, dv^*$$

This formula determines our

Smooth generating
sections of the continuous $\cong \mathcal{S}(V, S)$
field over V^*/Λ^* (Schwartz-class
functions)

When we analyze the Fourier isomorphism $C(\Lambda^*/\Gamma^*) \cong C^*(\Lambda)$ we find that the Hilbert $C^*(\Lambda)$ -module generated by the smooth sections above has a very simple form. The $C^*(\Lambda)$ -module structure is generated by the natural action of Λ on $\mathcal{S}(V, S)$ by translation on V . The $C^*(\Lambda)$ -valued inner product is given by

$$\langle f_1, f_2 \rangle(k) = \langle f_1, f_2^k \rangle_{L^2}.$$

On the right is the usual L^2 -inner product of functions, and

$$f^k(v) = f(v-k) \quad (v \in V, k \in \Lambda).$$

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The Dirac operators $D_{L,x}$ combine to form a single operator acting on smooth sections of the continuous field.

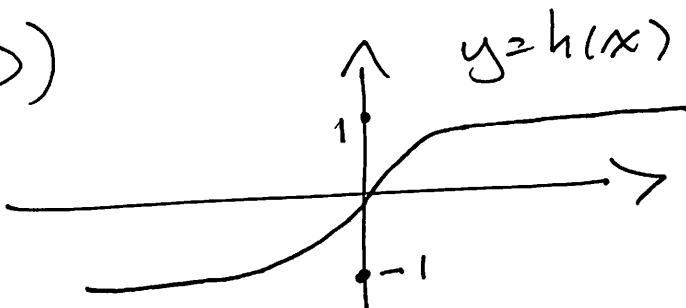
From the Schwartz-space perspective it is nothing but the Dirac operator on \mathbb{V} :

$$D: \mathcal{S}(\mathbb{V}, S) \longrightarrow \mathcal{S}(\mathbb{V}, S).$$

From this point of view the assembly map is given by a prescription that is a small variation of Atiyah's original idea for K-homology:

First, make D into a bounded operator (on our Hilbert $C^*(\Gamma)$ -module) by creating

$$F = h(D)$$



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Next, note that $C(V/\Lambda)$ acts as bounded operators on our Hilbert $C^*(\Lambda)$ module by pointwise multiplication of functions on V/Λ (viewed as periodic functions on V) with S -valued functions on V .

This action has the pseudolocality property that

$f \in C(V/\Lambda) \Rightarrow [F, f]$ is a compact operator on the Hilbert module.

Just as Atiyah did, we can construct from F and a class in $K(V/\Lambda)$ a Fredholm operator (in the Hilbert module sense*) and take its index

* This sense can be inferred from the correspondence between continuous fields and Hilbert modules. An operator is compact if it is a norm-limit of operators of the form $s \mapsto \sum_{i=1}^n t_i \langle s_i, s \rangle$.

We obtain a map

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$$K^*(V/\Lambda) \longrightarrow K_*(C^*\Lambda),$$

and this is the original assembly map.

The above can be repeated in much greater generality. To take a modest step in this direction, we can replace Λ by any π for which $B\pi$ can be modelled by a closed spin^c manifold M^* (playing the role of V/Λ). The smooth compactly supported spinors on the universal cover (playing the role of V) generate a Hilbert $C^*(\pi)$ -module and the Dirac operator on the

* Actually compactness of π is not necessary.

universal cover gives the assembly^{4.21} map for π .

Having taken all these steps, let us return to the "dual" assembly map

$$K^*(V^*/\Lambda^*) \rightarrow K^*(V/\Lambda)$$

that inverts the actual assembly map for Λ .

Of course it can be described by exactly the procedure we've just gone through... except with Λ replaced by Λ^* , etc.

But let us apply the Fourier Isomorphism

$$(A) \quad \mathcal{S}(V^*, S) \xrightarrow{\approx} \mathcal{S}(V, S)$$

↑
"dual" map
defined using
this

and take another look.

The left-hand side of (\star) carries
 (after a completion) a $V C^*(\Lambda^*)$ -module
 $\underset{\text{Hilbert}}{\text{structure.}}$
 On the right hand side
 there is (after completion) of course
 a corresponding Hilbert $C^*(\Lambda^*)$ -module
 structure. It can be defined very
 simply using the Fourier isomorphism.

$$C^*(\Lambda^*) \cong C(V/\Lambda).$$

Namely the module action of $C(V/\Lambda)$
 on $S(V, S)$ in (\star) is the obvious
 one, and the inner product is

$$\langle f_1, f_2 \rangle(v) = \sum_{k \in \Lambda} (f_1(v+k), f_2(v+k))_S,$$

where the round brackets $(,)_S$ denote
 the Hermitian inner product on the
 spinor vector space S .

The Dirac operator on $\mathcal{S}(V^*, S)$ becomes, under the isomorphism $(\#)$ the multiplication operator

$$(D.f)(v) = \int_1 c(v) f(v) \quad (f \in \mathcal{S}(V, S))$$

where $c(v)$ is Clifford multiplication.

Note that this is linear with respect to the $C(V/\Lambda)$ -module structure, as it must be.

The representation of $C(V^*/\Lambda^*)$ on the left hand side of $(\#)$ becomes, under the isomorphism $(\#)$ and the Fourier isomorphism

$$C^*(V^*/\Lambda^*) \cong C^*(\Lambda),$$

the natural action of Λ on $\mathcal{S}(V, S)$ by translations.

Having so reinterpreted the dual assembly map for Λ , we can ask if it makes any sense for more general groups π . It does.

Suppose again that $M = \mathbb{B}\pi$ is a closed spin^c-manifold. Let V be its universal cover and pull back the spinor bundle on M to obtain the π -equivariant bundle S on V . Consider

$$C_c(V, S).$$

Put a $C(M)$ -module structure on it by pointwise multiplication of functions against sections. Define a $C(M)$ -valued inner product by

$$\langle f_1, f_2 \rangle(m) = \sum_{g \in \pi} (f_1(g\tilde{m}), f_2(g\tilde{m}))_S,$$

where the sum is over the points in the preimage of m under $V \rightarrow M$.

Define a representation of C^*_π 4.25
using the action of π by deck
transformations.

What about the operator D ? For $\pi = \Lambda$
we took D to be a Clifford
multiplication operator. Here we do
the same, although we need an extra
geometric hypothesis to able to do so.

We therefore assume that $\pi = B\pi$ (which
for simplicity we are taking to be
a closed manifold) admits a metric
of non-positive sectional curvature.

Example : $\pi =$ uniform lattice in a semisimple
group. Take $B\pi = \pi \backslash G / K$.

The universal cover \tilde{V} is then a complete, simply connected manifold of non-positive sectional curvature — a so-called Hadamard manifold.

We define, following the case $\pi = 1$,

$$(\nabla f)(v) = \sum_i c(v) f(v) \quad (\text{f.e. } C_c^\infty(V, S))$$

where $c(v)$ is Clifford multiplication not by v (which is not a vector!) but by the value at $v \in V$ of the gradient of the function

$$v \mapsto \text{distance}(v_0, v)^2$$

Here v_0 is some (arbitrarily chosen) point in V .



This agrees with our previous definition of D in the case $\pi = \Lambda$.

We obtain

- a Fredholm operator F on a Hilbert $C(\beta\pi)$ module
- an action of $C^*(\pi)$ on this Hilbert module that commutes with F , up to compact operators.

and this gives

$$K_*(C^*\pi) \longrightarrow K_*(\beta\pi)$$

by Atiyah's method.

Theorem (Kasparov) The above map
is a left inverse of the assembly
map: the composition

$$K^*(B\pi) \rightarrow K_*(C_*^\pi) \longrightarrow K^q(B\pi)$$

is the identity.

Whether it is also a right inverse is
not generally known. But there is
one interesting case that can be
handled. It involves new difficulties
that require new techniques, needed
to cope with infinite dimensions.
But the general approach is the
one sketched above.

Definition A countable discrete group π is a-T-menable if there is an action

$$\pi \times H \longrightarrow H$$

on a Hilbert space such that

(a) each $g: H \rightarrow H$ ($g \in \pi$)

is an affine isometry and

(b) for every $v \in H$, $\lim_{g \rightarrow \infty} \|g \cdot v\| = 0$.

This definition (due to Gromov) covers an interesting range of examples:

- amenable groups
- Coxeter groups
- real & complex hyperbolic groups
- Thompson's group F
- CAT(0) groups
- etc

However not infinite property Γ
group is a- Γ -menable.

Theorem (Hjelg-Kasparov) If π is
a- Γ -menable, then the Baum-Connes
assembly map is an Isomorphism.

The additional strength of the conclusion
can be attributed to the following
simple geometric fact — every isometric
action on H is \mathbb{R}^n homotopic,
through isometric actions, to a
linear action (i.e. an action with
a global fixed point).