Lecture 3

Some applications of K-homology

I'll review some of what we've said about K-homology, supply a few analytic details, then quickly sketch some applications. I'll end with a potential application — an open problem on the quantization commutes with reduction phenomenon in symplectic geometry.

K-Homology & Asymptotics for tD

Recall Atiyah's cycles for the K-homology group $K_0(X)$: a cycle is a Fredholm operator

$$F : H_0 \longrightarrow H_1$$

between Hilbert spaces equipped with $C(X)$ module structures. We require that

$$f \in C(X) \Rightarrow [F, f] \text{ is a compact operator from } H_0 \text{ to } H_1.$$

The key example is the phase of D in its polar decomposition

$$D = F\,|D|.$$

Recall also Kasparov's equivalence relation: homotopy (through Fredholm operators on continuous fields).

(Incidentally there is a nearly identical theory for the odd K-homology group $K_1(X)$. Cycles are self-adjoint Fredholms

$$F : H \longrightarrow H$$

(on Hilbert spaces equipped with $C(X)$ actions, with $[F, f]$ compact; the equivalence relation is htpy. They give maps

$$K^*(X \times Y) \longrightarrow K^{*-1}(Y)$$

that are functorial & multiplicative in the Y-variable.)

Why bother with a concrete definition of K-homology (rather than an abstract one via Spanier-Whitehead duality, for example)?

The idea is that if we want to calculate some invariant I of an operator D (like its index, for example) we can try to find some D' for which $I(D')$ <u>can</u> be calculated, and then show D & D' define the same elt of $K_*(X)$ by an explicit homotopy. As long as I is defined on K-homology, we're done.

Before giving some examples I want to comment on features of elliptic operator theory that are behind the property

$$\boxed{f \in C(X) \Rightarrow [F, f] \text{ compact.}}$$

What are most relevant are these things:

(a) h C_0 function $\Rightarrow$ $h(D)$ compact (that is, D has compact resolvent)

(b) $\lim_{t \to 0} [h(tD), f] = 0$ for such C_0 functions h.

Here is a proof of (b). (The proof of (a) is more difficult. It uses Fourier theory, but interestingly it can be efficiently organized using the family $h(tD)$.) The h's for which (b) holds form an algebra (since the commutator operation is a derivation), so it suffices to consider the generators $h(x) = (x \pm i)^{-1}$. We get

$$[(tD \pm iI)^{-1}, f] = (tD \pm iI)^{-1} t [f, D] (tD \pm iI)^{-1}$$

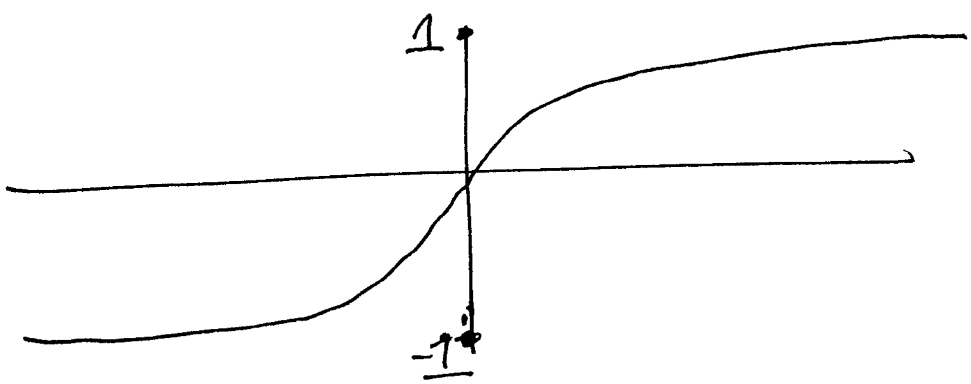
and (i) $[f, D]$ is bounded while (ii) $(tD \pm iI)^{-1}$ is uniformly bounded in t (by 1).

On to Atiyah's pseudolocality condition.

First

$$\begin{pmatrix} 0 & F^* \\ F & 0 \end{pmatrix} = h(D) + \text{compact}$$

where h is any continuous function as follows



We'll take

$$h(x) = \frac{2}{\pi} \arctan(x)$$

and study

$$[\arctan(D), f].$$

Well,

$$\arctan(D) = \int_0^1 \frac{d}{dt} \arctan(tD)\, dt$$

$$= \int_0^1 D(1+t^2D^2)^{-1}\, dt$$

Now

$$D(1+t^2D^2)^{-1} = \frac{1}{t}(tD+iI)^{-1} + \frac{1}{t}(tD-iI)^{-1}$$

so that

$$(\ast)\quad [\arctan(D), f] = \int_0^1 \frac{1}{t}[(tD\pm i)^{-1}, f]\, dt$$

(sum $\pm$ terms)

But

$$\left[\frac{1}{t}(tD\pm i)^{-1}, f\right] = (tD\pm i)^{-1}[f, D](tD\pm i)^{-1}$$

The integrand in $\ast$ is continuous on $t>0$, compact operator-valued, and $O(1)$ as $t \to 0$. The compactness of $(\ast)$ follows.

Toeplitz Theory

The first serious application of the $I(D) = I(D')$ philosophy, after the index theorem itself, is to Toeplitz operators. Consider

$\Omega \subseteq \mathbb{C}^n$ bounded strongly pseudoconvex domain

So roughly

$$\Omega = \{ r < 0 \}$$

where the defining function $r : \mathbb{C}^n \to \mathbb{R}$ satisfies

$$\left[\frac{\partial^2 r}{\partial z_i \partial \bar{z}_j} \right] > 0 \quad \text{(pos. def. matrix)}$$

Example — an actual convex Ω, like a ball.

The Bergman space $A^2(\Omega) \subseteq L^2(\Omega)$ is the Hilbert subspace of holomorphic functions.

<u>Theorem</u> The orthogonal projection P onto the Bergman space commutes with $f \in C(\bar{\Omega})$ (viewed as an operator on $L^2(\Omega)$) modulo compact operators.

This is a consequence of elliptic theory on Ω for the Dolbeault operator (with respect to a suitable Kähler metric) plus Gromov's beautiful theorem about Kahler mflds with $\omega = d(\text{bounded})$.

Easier: if $f \in C_c(\Omega)$ then $f \cdot P$ and $P \cdot f$ are individually compact.

We get a class

$$[\text{Bergman}] \in K_1(\partial\Omega)$$

from

$$F = P - P^{\perp} = 2P - I$$

(let $f \in C(\partial\Omega)$ act on $L^2(\Omega)$ by extending f to a tubular neighborhood of the boundary).

The class determines the indexes of Toeplitz operators

$$T_f = \begin{pmatrix} P & & \\ & \ddots & \\ & & P \end{pmatrix} f \begin{pmatrix} P & & \\ & \ddots & \\ & & P \end{pmatrix}$$

on $A^2 \oplus \cdots \oplus A^2$, where

$$f : \partial\Omega \to GL(k, \mathbb{C})$$

(note that f determines a class in $K^1(\partial\Omega)$, the K-theory of the boundary).

Baum-Douglas-Taylor proved that

(☆) $[\text{Bergman}] = [\text{Dirac}_{\partial\Omega}]$

and so reduced the Bergman index problem to Atiyah - Singer theory.

They proved (*) in an interesting way. They considered

$$K_0(\Omega) \xrightarrow[\substack{\text{K-homology} \\ \text{long exact} \\ \text{sequence.}}]{b = \text{boundary in}} K_1(\partial\Omega)$$

One has

$$b\,[\mathrm{Dirac}_{\Omega}] = [\mathrm{Dirac}_{\partial\Omega}]$$

always. The found $[D'] \in K_0(\Omega)$ so that

① $b\,[D'] = [\mathrm{Bergman}]$

② $[D'] = [\mathrm{Dirac}_{\Omega}]$.

A good choice for D' is the same Dolbeault operator for a certain complete

Kahler metric that was mentioned before. Item ① follows from Gromov's theorem again, while item ② follows from a general fact that different Dirac operators for different metrics (but the same spin^c structure) all give the same <u>fundamental class</u>

$$[M] \in K_n(M) \qquad n = \dim(M)$$

for a spin^c manifold M.

Geometric K-Homology

Let's continue with Dirac-type operators.

There is a geometric model for K-homology, due to Paul Baum, ~~that~~ deals directly with Dirac operators on spin^c manifolds and involves no functional analysis.

$$K_0^{geom}(X) = \left\{ \begin{matrix} \text{geometric} \\ \text{cycles on } X \end{matrix} \right\} \Big/ \text{equivalence}$$

Geometric cycles are triples

$$(M, E, f)$$

where

M = even-dimensional closed spin^c manifold

E = complex vector bundle on M

$f : M \longrightarrow X$

Such a cycle determines an element of the Atiyah-Kasparov K-homology group. We form

$$D_M = \text{Dirac operator on } M$$

$$D_{M,E} = D_M \text{ coupled to } E$$

(so $D_{M,E}$ acts on spinors tensor sections of E)

From $D_{M,E}$ we extract the phase, or equivalently construct $F = h(D_{M,E})$, and we obtain a class

$$[M,E] = [F] \in K_0(M)$$

Then we use f to push forward to X: we form

$$[M,E,f] = f_*[M,E] \in K_0(X)$$

<u>Remark</u> $K_0(M)$ is a module over the K-theory ring $K(M)$, and we could equivalently write

$$[M,E,f] = f_*([E] \cap [M]) \in K_0(X)$$

(The odd group $K_1^{geom}(X)$ is defined in the same way, using odd-dimensional M. The above formula still applies.)

The main theorem about $K^{geom}(X)$ is that the prescription above determines a natural

$$Q : K_*^{geom}(X) \to K_*(X)$$

that is an <u>isomorphism</u> (for reasonable X, say those with the htpy type of a finite CW complex)*

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* On general compact metric X the Atiyah-Kasp. theory satisfies the <u>Steenrod axioms</u>; the geometric theory does not.

The well-definedness of Q is a form of the index theorem, in the spirit of Grothendieck Riemann-Roch. Consider for example

$$\begin{tikzcd} K_0^{geom}(X) \arrow[r, "Q"] \arrow[d] & K_0(X) \arrow[d] \\ K_0^{geom}(pt) \arrow[r, "Q"'] & K_0(pt) \end{tikzcd}.$$

Going this way $\overrightarrow{\ \ }\!\downarrow$ associates to (M, E, f) the analytic index of $D_{M,E}$, while the map $\downarrow\!\!\longrightarrow$ (which really involves no Fredholm operators) can easily be identified (using the machinery of topological K-theory and/or cohomology) with the topological index.

A Remark about the geometric equiv. relation

Baum's relation serves a purpose that is close to opposite that of Kasparov's. It incorporates as few and as elementary a set of steps as possible, with a view to making it as easy as possible to define invariants I that are well defined at the level of K-homology.

One example is the "invariant" Q that attaches to a cycle its K-homology class à la Atiyah-Kasparov. A special instance of this is the analytic (i.e. Fredholm) index.

Here is another example coming from secondary invariants. We'll consider $X = B\pi$ for some discrete group π.

Suppose given two finite-dimensional representations

$$\sigma_1, \sigma_2 : \pi \longrightarrow U(N)$$

Attached to each is a flat Hermitian bundle on $B\pi$. Given a cycle (M, E, f) we can pull back the flat bundles to M along f, then couple the Dirac-type operator $D_{M,E}$ to each. We get two Dirac-type operators that we'll just call D_1 and D_2.

Associated to each is an eta-invariant, which is a renormalized count of the number of positive eigenvalues of the operator, minus the number of negative eigenvalues (unrenormalized, this is $\infty - \infty$; renormalized, we could in principle get any real number).

We're interested in the difference

$$\rho_{12}(D) = \eta(D_1) - \eta(D_2),$$

which is called a <u>relative eta invariant</u>.

While each of $\eta(D_1)$ and $\eta(D_2)$ is a delicate thing (it's hard to prove the renormalizations even exist), the difference defines a map

$$\rho_{12} : K_1^{geom}(B\pi) \longrightarrow \mathbb{R}/\mathbb{Z}.$$

(The invariant is defined on the even group K_0 but vanishes thanks to supersymmetry.)

Checking ρ_{12} is well-defined is fairly straightforward using Baum's relation. The Atiyah-Kasparov side of the story is another matter.

However we can use the fact that Q is an isomorphism to transfer ρ_{12} to analytic K-homology. There we can apply C^*-algebraic techniques coming from the Novikov and Baum-Connes conjectures (to be discussed in the next lecture).

Here's one application (a theorem discovered using other means by Weinberger):

Theorem The relative eta invariants of the signature operator* are oriented homotopy invariants, mod $\mathbb{Q}$.
(For more, see the paper "Relative eta invariants and K-homology" by John Roe & myself, on my web page.)

*This is the Dirac-type operator whose index is the signature of an oriented closed manifold.

World's Simplest Index Theorem

(see Erik van Erp, Annals of Math, to appear)

(Truly the world's simplest index theorem is the one for Toeplitz operators on the circle. The theorem to be discussed is the world's simplest-to-state-but-still-difficult index theorem.)

Suppose M is closed, 3-dimensional, that (a) $H \subseteq TM$ is a 2-plane bundle, that (b) T/H is trivial, and that (c) the brackets $[H, H]$ span T/H. Every H satisfying (a) & (b) can be maneuvered into one also satisfying (c).

Suppose T/H is trivialized by a nowhere vanishing vector field Z on M. This trivialization and the Lie bracket give ~~a~~ symplectic structures on the fibers of H, and hence an orientation.

Put a metric on H and let Δ_H be the corresponding <u>sublaplacian</u> on functions:

$$\langle \Delta_H u, v \rangle = \langle \mathrm{grad}_H u, \mathrm{grad}_H v \rangle$$

where $\mathrm{grad}_H u$ is the section of H with

$$\langle \mathrm{grad}_H u, X \rangle = X(u) \quad (X \in H).$$

Now consider

$$D = \Delta_H + i\alpha Z$$

where $\alpha : M \longrightarrow \mathbb{C}$.

<u>Theorem</u> As long as

$$\mathrm{range}(\alpha) \cap \{\text{odd integers}\} = \phi$$

the operator D is Fredholm. (!)

Moreover the index has the same sort of stability properties as in the usual elliptic world (we can calculate it using smooth fns, or L^2-fns, or distributions, etc).

<u>Index Theorem</u> Suppose $2C \in H_1(M)$ is Poincaré dual to the euler class of H. Then

$$\mathrm{Index}(D) = \sum_{\substack{k \text{ odd} \\ k>0}} (k-1)\, \mathrm{WindingNumber}_C\left(\frac{\alpha-k}{\alpha+k}\right)$$

This sort of formula is due to Melrose & Epstein; van Erp puts it into a very general context.

Some of the ingredients for van Erp's theory:

- The H-tangent bundle of M. The fibers are not linear spaces (vector groups) but the <u>Heisenberg groups</u>

$$G_m = H_m \oplus T_m / H_m$$

- Deformation space for $M \hookrightarrow M \times M$ appropriate to the H-tangent bundle.

- The natural deformation from TM to $T_H M$. (Rescale by replacing Z with $t^{-1} Z$, or, if you like, by replacing the symplectic forms ω_m on the H_m by $t\omega_m$.)

These deformations connect the index problem for D to an index problem for any ordinary elliptic (pseudo) differential operator on M.

Quantization Commutes with Reduction

This is a fairly general principle in index theory. It holds for Kähler manifolds, symplectic manifolds, and in various further degrees of generality that are harder to pin down. I'll describe the case of Kähler manifolds, then indicate how K-homology might help clarify the phenomenon in more general contexts.

Suppose that M is a closed Kähler manifold. In particular its tangent bundle carries a complex Hermitian structure:

$$J : TM \to TM$$
$$J^2 = -I$$

$$h(X,Y) = g(X,Y) - \sqrt{-1}\,\omega(X,Y)$$

hermitian metric — Riemannian metric — Real 2-form

We have

$$\omega(X,Y) = g(JX,Y).$$

The Kahler condition says that $d\omega = 0$. We'll suppose that M is <u>prequantizable</u>, which means there is a Hermitian line bundle L with connection ∇ such that

$$\nabla^2 = \sqrt{-1}\,\omega.$$

Now suppose a compact group G acts on the whole package...

The infinitesimal action of G defines a derivative operator

$$D_X : \Gamma(L) \longrightarrow \Gamma(L)$$

for $X \in \mathfrak{g}$. Thinking of X as a Killing vector field we also have

$$\nabla_X : \Gamma(L) \longrightarrow \Gamma(L)$$

Because both D_X & ∇_X obey the Leibniz law and are compatible with the Hermitian structure on L,

$$D_X = \nabla_X + \sqrt{-1}\,\mu_X$$

for some function

$$\mu_X : M \longrightarrow \mathbb{R}.$$

The compatibility of ∇ with ω gives

$$\omega(X,Y) + Y(\mu_X) = 0$$

for all $X \in \mathfrak{g}$ and all tangent vectors Y. Equivalently

$$J \cdot \mathrm{grad}(\mu_X) = X$$

So although X is not a gradient vector field, its rotation by $90°$ is.

Assembling all the μ_X together gives

$$\mu : M \longrightarrow \mathfrak{g}^*.$$

This is the <u>moment map</u>.

We shall assume that

$$\boxed{0 \text{ is a regular value of the moment map}}$$

Then $\mu^{-1}(0)$ is a manifold and

$$M \approx \mu^{-1}(0) \times \mathfrak{g}^*$$

near $\mu^{-1}(0)$.

The relation $J \cdot \operatorname{grad} \mu_X = X$ implies G acts locally freely on $\mu^{-1}(0)$. For simplicity we'll assume the action is actually free. Then

$$M /\!\!/ G := \mu^{-1}(0)/G$$

is a manifold, called the <u>reduction</u> of M. It is actually a symplectic manifold since $\omega\big|_{\mu^{-1}(0)}$ is basic and so is pulled back from a symplectic form on $M//G$. In fact $M//G$ is Kahler, and moreover the line bundle L (and connection ∇) descend to $M//G$.

<u>Theorem</u> ($[Q,R]=0$)

$$\mathrm{Index}\left(\mathrm{Dolbeault}_{M//G,\, L//G}\right)$$

$=$ Multiplicity of the trivial representation of G in $\mathrm{Index}(\mathrm{Dolbeault}_{M,L})$.

The proof was started by Guillemin–Sternberg and finished (actually in the symplectic context) by Meinrenken.

There are now various proofs, but I won't describe any of them.

Instead I'll describe a problem — to set $[Q,R]=0$ into the context of K-homology.

This is based on the observations that

① Both analytic (Atiyah-Kasparov) and geometric (Baum) K-homology have natural G-equivariant counterparts for a compact group (just make the cycles and the ~~rel~~ relations G-equivariant).

② There is a natural reduction map

$$R: K_*^G(X) \longrightarrow K_*(X/G)$$

in analytic K-homology that takes

a cycle $F: H_0 \longrightarrow H_1$ to

$$F\big|_{H_0^G} : H_0^G \longrightarrow H_1^G .$$

Note that if $X = pt$ this takes F to the multiplicity of the trivial representation of G in its index (this is an integer and $K_0(pt) = \mathbb{Z}$).

<u>Problem</u> Define

$$R^{geom} : K_*^{geom, G}(X) \longrightarrow K_*^{geom}(X/G)$$

so that $[Q, R] = 0$: the diagram

$$\begin{tikzcd} K_*^{geom,G}(X) \arrow[r, "R^{geom}"] \arrow[d, "Q"'] & K_*^{geom}(X/G) \arrow[d, "Q"] \\ K_*^G(X) \arrow[r] & K_*(X/G) \end{tikzcd}$$

commutes.

Obviously, we want it to be clear that

$$R^{geom}(M, L, f) = (M/\!/G, L/\!/G, f/G)$$

in the "classical" Kahler or symplectic case.