

## Lecture 2

### The Index Theorem

First, more on continuous fields.

Here's the definition again: a continuous field over  $X$  is

- 1) A family  $H = \{H_x\}$  of Hilbert spaces and
- 2) A vector space  $\Gamma(H)$  of sections

such that

a) If  $s \in \Gamma(H)$  then  $x \mapsto \|s(x)\|$  is continuous.

b) The values  $\{s(x) : s \in \Gamma(H)\}$  are dense in  $H_x$ .

c) Let  $t$  be any section. Suppose that  $\forall x \in X, \exists \varepsilon > 0$ . There is a nbhd  $U$  of  $x$  in  $X$  and a section  $s \in \Gamma(H)$  such that

$$\sup_{y \in U} \|s(y) - t(y)\| < \varepsilon$$

Then  $t \in \Gamma(H)$ .

The elements of  $\Gamma(H)$  are called the continuous sections of  $H$ .

Given a family of Hilbert spaces  $H = \{H_x\}$ , and a vector space of sections satisfying (a) & (b), there is a unique enlargement of the family that also satisfies ~~that also~~ continuous fields specified this way.

A geometric example: given a submersion  $M \rightarrow X$  of smooth manifolds with a smooth family of smooth measures\* on the fibers  $\mathcal{M}_x$ , we can form

$$H_x = L^2(\mathcal{M}_x)$$

and generate a continuous field from the smooth, compactly supported functions on  $M$ .

Before exploring this further, let's discuss again some abstract points. . .

\*We could skip the measure part by using  $\frac{1}{2}$ -densities

Examples of continuous fields: Constant fields have all  $H_x$  equal to a fixed Hilbert space; continuous sections are the continuous functions from  $X$  into this Hilbert space.

Trivial fields are by definition those that are isomorphic to constant fields

Continuous fields can be restricted to arbitrary subspaces (or more generally pulled back along continuous maps). So for example, one can talk about locally trivial fields. Not every continuous field is locally trivial.

If  $Y$  is an open set in a compact  $X$  (say) and if  $H$  is a continuous field on  $Y$ , we can form its push-forward to  $X$ :

- $H_x = 0 \quad \text{if } x \notin Y$
- Continuous sections over  $X$  we have over  $Y$  such that if  $y \rightarrow \infty$  then  $\|s(y)\| \rightarrow 0$ .

Pushforwards don't have constant or locally constant fiber dimension, so they aren't locally trivial.

A Hermitian vector bundle determines a continuous field (the fibers are the fibers; the continuous sections are the continuous vectors).

Every Hermitian vector bundle (over a compact space) is a direct summand of a trivial bundle of finite rank. There is a beautiful generalization of this: every (countably generated) continuous field (over a locally compact space) is a summand of a trivial continuous field. This is the stabilization theorem.

So continuous fields are the infinite-dimensional "projective" fields. But note that by an Eilenberg swindle we can always write

$$H \oplus \text{trivial} = \text{trivial}.$$

Hence there is no interesting K-theory group here, as things stand just yet...

To get an interesting K-theory one needs to consider Fredholm operators between C\*-fields,

$$F: H_0 \rightarrow H_1,$$

as we discussed last time.

Kasparov's version of the Atiyah-Singer theorem says that the set of homotopy classes of Fredholm operators between C\*-fields over  $X$  (locally cpt) is Atiyah-Hirzebruch K-theory.

## Dirac Operators

Let's now discuss how to fit Dirac operators into the context of Hilbert spaces of C\*-fields of Hilbert spaces. We'll start with single operators, not families.

It's convenient to introduce  $(\mathbb{Z}/2\mathbb{Z})$  gradings and work with

$$\begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix} \quad (\text{here } D^* = \text{formal adjoint})$$

in place of  $D$ . We'll call the whole thing  $\mathcal{D}$  from now on, and set

$$\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The original  $D$  can obviously be recovered from the new  $D$  together with  $\Sigma$ .

The value of the change is that the new  $D$  is essentially self-adjoint (if the underlying manifold on which it is defined is closed).

This means we can develop a functional calculus for  $D$ . In the present context this assumes a concrete form: there is an orthonormal basis of smooth sections of the bundle on which  $D$  acts with

$$D u_n = \lambda_n u_n$$

$$\lambda_n \in \mathbb{R}, |\lambda_n| \rightarrow \infty$$

If  $h$  is a bounded (continuous) function on  $\mathbb{R}$  we set

$$h(D)u_n = h(\lambda_n)u_n,$$

which defines a bounded operator.

If  $h$  is a  $C_0$ -function (vanishing at  $\pm\infty$ ) then  $h(D)$  is a compact operator.

The eigenvalue/eigenvector perspective is concrete and helpful to get started, but a more abstract, functional-analytic approach is in many respects preferable. For example it leads quickly to families versions. Here is the result:

Theorem Given a submersion  $M \xrightarrow{*} X$  and a smooth family of Dirac type operators on the ~~fiber~~ fibers  $M_x$ ,

(1) If  $\square$  is compact then the family of operators  $h(D) = \{h(D_x)\}$

is a bounded operator on the  $C^k$  field associated to the submersion, and it is a compact operator if  $h$  is a  $C^\infty$ -function.

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\*We allow  $X$  to have boundary but not the fibers  $M_x$ . So the submersion is weakly of the form  $Y \times X \rightarrow X$  where  $X$  might have boundaries but  $Y$  does not.

2) If  $\Pi$  is noncompact then  $h(D)$  is still a bounded operator, as long as each  $D_x$  is essentially self-adjoint (so that  $h(D_x)$  is defined).

If  $h$  is a  $C_0$ -function on  $\mathbb{R}$  and if  $f$  is a  $C_0$ -function on  $\Pi$ , then the product.

$$f \cdot h(D)$$

is compact.

Remark Of course, it is the abstract perspective (basically the Fourier transform and the theory of constant coefficient operators, plus Hilbert's spectral theorem) that lead to the eigenvalue picture in the first place. My point is that those techniques are, first of all, very naturally situated within the context of Hilbert space, and that the techniques extend without additional

effort to families.

Let me say a bit more about the issue of  $h(D)$  being noncompact when  $M$  is not compact.

Actually it's convenient to work in a slightly more abstract context. We'll suppose:

(a)  $X$  is a locally compact space.

(b)  $D = \{D_x\}$  is a family of essentially self-adjoint operators on the fibers of a continuous field over  $X$ .

(c) If  $h$  is a co-function then  $h(D) = \{h(D_x)\}$  is a bounded operator on the field.

In examples, the sort of continuity that is implicit in the last assumption (that  $h(D)$  maps continuous sections to continuous sections) is easy to achieve.

Remark By the way, a Hilbert space operator (perhaps unbounded and only densely defined) is essentially self-adjoint if

$$\textcircled{1} \quad \langle Du, v \rangle = \langle u, Dv \rangle$$

This says that  $D$  is formally self-adjoint. It implies that the operators  $D \pm iI$  are 1-1 and in fact bounded below.

\textcircled{2}  $(D \pm iI)$  have dense range.

This implies that the operators  $(D \pm iI)^{-1}$  — which are bounded by \textcircled{1} — are densely defined and hence extend to bounded Hilbert space operators.

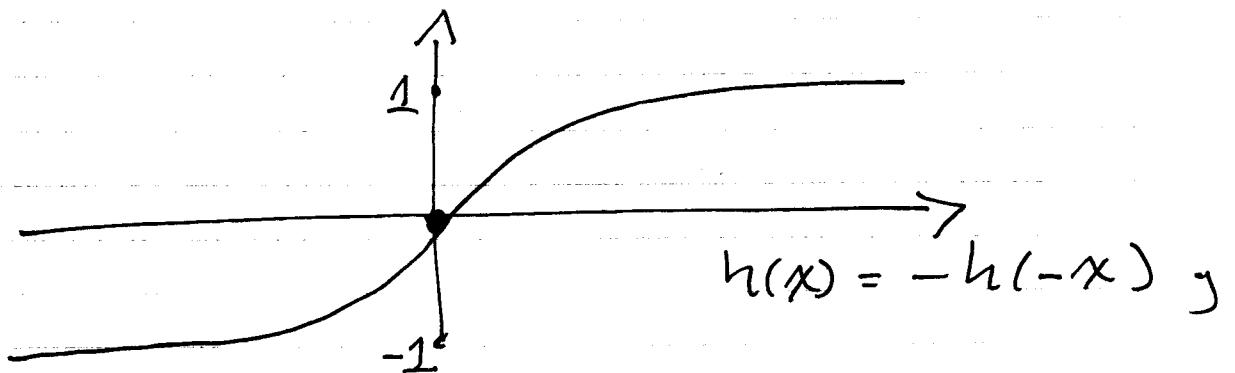
The functional calculus begins by defining

$$h \pm (D) := (D \pm iI)^{-1}$$

If  $h \pm (x) = (x \pm i)^{-1}$ . Functional analysis guarantees that this prescription extends uniquely to a homomorphism from bounded continuous functions  $h$  on  $\mathbb{R}$  to bounded operators.

Exercise If  $D$  is bounded and if \textcircled{1} holds then \textcircled{2} does too.

Why are we interested in  $h(D)$  being compact when  $h$  is a co-function? For this reason: if we take  $h$  to be not a co-function but one like this



for example

$$h(x) = \frac{2}{\pi} \arctan(x)$$

(which is a happy choice for many purposes) then  $h(D)$  looks like

$$h(D) = \begin{pmatrix} 0 & F^* \\ F & 0 \end{pmatrix}$$

and  $F$  is a Fredholm operator—in fact  $F$  is isomorphic equal to the polar decomposition  $F$  for  $D$  (see Lecture 1), up to a compact operator.

If  $h(D) = \text{compact}$  for all  $\zeta \in h$   
then we say  $D$  has compact  
resolvent. (Because the compactness  
condition in fact only need be  
checked on  $h(x) = (x+i)^{-1}$ ). Using  
this terminology

Compact resolvent  $\Rightarrow$  Fredholm.

Here are some examples

- Dirac operators on the fibers of a compact submersion. This we already discussed
- Suppose  $V$  is a  $SU(n)$  vector bundle of even rank  $2k$  over a compact space  $X$ .  
This means there is a Hermitian bundle  $S$  over  $X$  and an isomorphism of ( $\mathbb{Z}/2$  graded)  $*$ -algebra bundles  $\text{Cliff}(V) \cong \text{End}(S)$   
over  $X$  (see any introductory account of Clifford algebras & K-theory).

Pull back  $S$  (a bundle over  $X$ ) to  $V$  and think of the result as being a continuous field of Hilbert spaces over  $V$ .  
The formula

$$E_V = \text{Cliford multiplication} \times \\ (\text{times } F_i)$$

gives a compact resolvent operator.

Remark In this case the unboundedness or non-compact resolvent nature of individual  $C_V$  is not an issue - It is the noncompactness of  $V$ , viewed as the parameter space, that needs to be handled.

And now here is an example of the last example. A spin<sup>c</sup> structure on a mfld  $M$  is a spin<sup>c</sup> structure on  $TM$ , as above. Suppose  $M$  is embedded in  $V = \mathbb{R}^N$  for some even  $N$ . Then the normal bundle

$$N_V M = V \times M / TM$$

carries a natural spin<sup>c</sup> structure, determined by the one on  $M$ .

The index of the Fredholm operator determined by the construction above is a class

$$\text{Th}(N_{\sqrt{t}}) \in K(N, M)$$

( $\text{Th}$  = "Thom class"). Think of  $N_{\sqrt{t}}$  as a tubular neighborhood in  $R^M$ , and push forward the K-theory class (equivalently, push forward the continuous field to  $R^M$ , then take the index). We get

$$\text{c}_*(\text{Th}(N_{\sqrt{t}})) \in K(R^M)$$

and by Bott periodicity

$$K(R^M) \cong \mathbb{Z}.$$

The integer we get is the topological index of  $M$ .

To summarize the last couple of pages, the topological index of Atiyah & Singer fits comfortably into the contexts of continuous fields, Fredholm indexes, etc.

We want to look at the analytic index the same way. Then we want to take advantage of our similar handling of the analytic & topological indexes to prove the index theorem of Atiyah & Singer (for Dirac ops).

The Dirac operator on  $\mathbb{M}$  is of course a single Fredholm operator. Here's how to get a family.

Let  $S$  be the spin module for  $V = \mathbb{R}^n$ . Define a ~~real~~ continuous field and operator on the field as follows

$$H_v \equiv L^2(\mathbb{M}, S_{\mathbb{M}}) \hat{\otimes} S \quad (\text{constant field})$$

Here  $S_{\mathbb{M}}$  = spinor bundle for  $\mathbb{M}$ .

$$D_v := \text{Dirac}_{\mathbb{M}} \hat{\otimes} I + I \hat{\otimes} C_v,$$

where  $C_v = \sqrt{-1} \times \text{Clifford mult by } v$ .

The index theorem is the assertion that they are equal. How to prove it?

We're going to build

(1) a space  $H_{V,M}$  that interpolates between  $N_{V,M}$  and  $V$

and

(2) a Fredholm operator on a continuous field over this space that interpolates between the Fredholm operators above that gave

$\text{Th}(N_{V,M})$  and  $\text{Index}(\text{Dirac})(\text{Bott},)$

The space  $H_{V,M}$  is the deformation space or deformation to the normal cone associated to  $M \hookrightarrow V$ .

As a set

$$H_{V,M} = N_{V,M} \times \{0\} \sqcup \bigsqcup_{t \in (0,1]} V \times \{t\}$$

$$= N_{V,M} \times \{0\} \sqcup V \times (0,1]$$

This carries a smooth  $(N+1)$ -dim'l

manifold structure so that

- $V \times (0,1]$  is an open subset
- The map  $N_{V,M} \rightarrow V \times [0,1]$  that is the identity on  $V \times (0,1]$  and the map

$$N_{V,M} \times \{0\} \xrightarrow{\text{proj}} M \times \{0\} \xrightarrow{\text{incl.}} V \times \{0\}$$

at  $t = 0$  is smooth.

- If  $f: V \rightarrow \mathbb{R}$  is a smooth function that vanishes on  $M$ , then

$$\tilde{f}: N_{V,M} \longrightarrow \mathbb{R}$$

$$\tilde{f}(v,t) = \frac{1}{t} f(v) \quad t \neq 0$$

$$f(X,0) = X(f) \quad t = 0$$

defines a smooth function on  $N_{V,M}$

- If  $x_1, \dots, x_p, y_1, \dots, y_q$  are local coordinates near a point of  $M$  such that  $x_1, \dots, x_p$  are local coordinates for  $M$  and  $y_1 = \dots = y_q = 0$  defines a manifold structure on  $N_{V,M}$ .

then  $x_1, \dots, x_q, \tilde{x}_1, \dots, \tilde{x}_q$  are local coordinates on  $N_{V\cap T}$  near  ~~$p$~~   $(p, 0) \in N_{V\cap T}$ .

There is a submersion

$$M \times V \times [0, 1]$$

$$\downarrow \\ N_{V \cap M}$$

defined by

$$(m, v, t) \quad (m, v, 0)$$

$$\downarrow t \neq 0$$

$$\downarrow t = 0$$

$$(m + tv, t)$$

$$(P_m(v), 0)$$

where  $P_m: V \rightarrow N_{V \cap m}$  is the projection from vectors in  $V$  to normal vectors at  $m \in M$ .

Build the continuous field whose fibers are the  $L^2$ -sections on the fibers of this submersion of the spinor bundle on  $M$  (pulled back to  $M \times V \times [0, 1]$ ) tensor the

spinor vector space for  $V = \mathbb{R}^N$ .

Let  $D$  be the fibrewise Dirac operator. Note that at  $t=0$  the fibers are the tangent spaces  $\{m\} \times T_m M \cong M \times V \times \{0\}$ , and the fiber Dirac operators are constant coefficient operators.

Then form

$$D \otimes I + I \otimes C$$

where  $C = \text{Clifford multiplication}$   
by  $v \in V$  on  $S$ , at the point  
 $(m, v, t) \in \Pi \times V \times [0, 1]$ .

This is the required Fredholm family,  
and its existence proves the index  
formula by connecting (geometrically)  
the topological & analytic inferences.

There is one calculation to be  
done. At  $t=0$  we have a family  
of partial differential operators  
rather than a ~~fixed~~ family of  
endomorphisms of finite dimensional

vector spaces, as in the definition of  $\text{Th}(N, M)$ . How are the two related?

The family of PDOs decomposes as a tensor product over  $t=0$ ,

$$Q \hat{\otimes} I + I \hat{\otimes} E$$

tensor product of

on the  $L^2$  fields  $\{L^2(TM_m, \text{End}(S_m))\}_{m \in M}$

and  $S_N$ , the spinor bundle of the normal bundle. Here  $E$  defines the Thom class, as before, while  $Q$  is the (square root of the) harmonic oscillator

$$Q_m = \text{Dirac}_{TM_m} \hat{\otimes} I + I \hat{\otimes} C$$

on  $\text{End}(S_m) \cong S_m \hat{\otimes} S_m^*$  valued functions, where  $C$  is right Clifford multiplication by  $X \in T_m M$  on  $\text{End}(S_m)$ .

As is well known, this has index 1 (even as a family) and since the  $Q$  is horotopic as a Fredholm family to its index, we obtain

at  $t = 0$  satisfy the Thom class  
for  $N\sqrt{t}$ , as required.

Further reading — See the paper  
"Tangent groupoid and the index  
theorem" on my webpage (to  
appear in a volume for Connes'  
GOth birthday). Don't be upset  
by the title: there are no groupoids  
in the paper!