Appendix to Lecture 1

Introduction to Continuous Fields of Hilbert Spaces

[These are notes from a lecture elsewhere, slightly reworked for the present purpose. Some of the items below are covered again in Lecture 2.]

X = topological space

$\{H_x\}_{x \in X}$ = family of Hilbert spaces

Defn A section of the family is an element of $\prod_{x \in X} H_x$, that is, a function s with $s(x) \in H_x$.

Refs
- Dixmier — Ch 10 of C*-algebra book
- Dixmier & Douady (1963)
- Mingo & Phillips (1984)
- Atiyah (Alg. top. & ops in Hilbert space 1969

<u>Defn</u> (Godement) A continuous field of Hilbert spaces over X is a family of Hilbert spaces $H = \{H_x\}_{x \in X}$ and a linear space of sections, $\Gamma(H)$, such that

(a) For each x, $\{s(x) \mid s \in \Gamma(H)\}$ is dense in H_x

(b) For each $s \in \Gamma(H)$, the function $x \mapsto \|s(x)\|^2$ is continuous

(c) If s' is <u>any</u> section and if for every $x \in X$, every $\varepsilon > 0$ there is some $s \in \Gamma(H)$ with

$$\| s(y) - s'(y) \| < \varepsilon$$

for all y near x, then $s' \in \Gamma(H)$

Examples • The constant field

• The continuous sections of a hermitian vector bundle

• Direct sums of continuous fields.

Defn An isomorphism from one cts field to another (over the same X) is a family of unitary isomorphisms of the fibers,

$$U_x : H_x \xrightarrow[\cong]{} H'_x$$

that carries $\Gamma(H)$ onto $\Gamma(H')$.

<u>Defn</u> If Y is a subspace of X, and H is a cts field on X, we define a cts field $H|_Y$ on Y by

$s' \in \Gamma(H|_Y) \iff$ for every $y \in Y$ and every $\varepsilon > 0$ there is a section $s \in \Gamma(H)$ such that

$$\| s(z) - s'(z) \| < \varepsilon$$

for all z near y in Y.

<u>Defn</u> A field is <u>trivial</u> if it is isomorphic to a cts field. This occurs, iff it has a <u>framing</u>, that is, ~~ortho~~ ~~orthonormal~~

a family of CTS sections that is, at each pt $x \in H$, an orthonormal basis for H_x.

<u>Defn</u> A field is locally trivial if there is an open cover $\{U_\alpha\}$ of X with $H|_{U_\alpha}$ trivial.

Of course, hermitian vector bundles give locally trivial fields. * Next page

Local triviality is not guaranteed! The following construction ~~a~~ provides an obvious obstruction.

<u>Defn</u> Let U be an open subset of a compact ~~space~~ space X (by the way, all spaces in this lecture

*Remark

Not every continuous field is locally trivial. One obstruction is variation in fibre dimension — to be discussed in a moment. But it is not the only one. Consider

$$X = \mathbb{C}P^1 \times \mathbb{C}P^1 \times \cdots$$

$$H = L \boxplus L \boxplus L \boxplus$$

the ("external") direct sum of the canonical bundles on the $\mathbb{C}P^1$-copies. It is not loc. triv (a char. class calculation. See Dix-Douady.

are assumed to be Hausdorff) and let H be a continuous field on U. Its <u>extension</u> <u>by</u> <u>zero</u> is the continuous field on X defined as follows

- $H_x = 0$ if $x \notin U$
- A section ~~s~~ is continuous iff its restriction to U is continuous, and $x \mapsto \|s(x)\|$ is continuous on X.

That is,

$$\lim_{\substack{u \to \infty \\ u \in U}} \|s(u)\| = 0 .$$

The fiber dimension of the extended field is not locally constant (arbitrarily?) and so the field is not locally trivial.

Simplest example:

$$X = [0,1]$$

$$U = (0,1]$$

$$H = \text{triv rank-one field on } U$$

$$\Gamma(\iota_* H) \cong \text{cts functions on } [0,1]$$
$$\text{~~that~~ vanish at } 0 \in [0,1].$$

It is also interesting and useful to push forward vector bundles on a general $U \subseteq X$ to continuous fields (not vector bundles anymore!) on X.

Let's analyze the simplest example in a bit more detail.

<u>Defn</u> A bounded (adjointable) operator from one field H_0 over X to another, H_1, is a uniformly bounded family of operators

$$T_x : H_{0x} \longrightarrow H_{1x}$$

~~such that~~
that carries continuous sections to continuous sections, and whose adjoint family

$$T_x^* : H_{1x} \longrightarrow H_{0x}$$

does the same.

The latter is not a given.
(<u>Example</u>... of operators on a trivial field.)

We can carry the usual operator terms (adjoint, isometry, projection, etc) over bounded operators between fields.

~~Now the ...~~

According to our definitions, if one field embeds isometrically in another, $H_0 \hookrightarrow H_1$, then we have

$$H_1 \cong H_0 \oplus H_{01}.$$

<u>Warning</u>: In general the (pointwise) orthogonal complement of a continuous field is not in general a continuous field.

But for the simple example of C^* Hconsidered earlier...

Lemma There is an isometric complemented embedding of H_0 into a trivial field.

Proof Pick a partition of unity for $(0,1]$ and define

$$V_x : H_x \longrightarrow \ell^2(\mathbb{N})$$

$$V_x \lambda = (\varphi_1^{1/2}(x)\lambda, \varphi_2^{1/2}(x)\lambda, \ldots$$

We have

$$V_x^* (\mu_1, \mu_2, \ldots)$$

$$= \varphi_1^{1/2}(x)\mu_1 + \varphi_2^{1/2}(x)\mu_2 + \cdots$$

and $V_x^* V_x = I_x$.

Contrast this with what occurs for vector bundles, ~~&~~ or if you like for finite-dimensional trivial fields. A direct summand of a finite dimensional trivial field certainly is locally trivial.

~~Ba~~ In fact the embedding phenomenon is general, at least for "countably generated" fields...

<u>Definition</u> A field is <u>countably generated</u> if there is a countable family of sections whose values in each fiber span the fiber.

Theorem (Dixmier-Douady)

Every countably generated field is iso to a summand of a trivial field.

Proof Let H be generated by $s_1, s_2, \dots$ and let H' be trivial with orthonormal basis $e_1 \; e_2 \; e_3$.

Let $w_1 \; w_2 \; w_3 \; \dots$ be a sequence of sections of H' in which each s_j occurs inf. often. Consider

$$v_j := 2^{-j} e_j \oplus w_j$$

a section of $H' \oplus H$. Take

The V_j <u>span</u> $H' \oplus H$

(meaning ~~H~~ the values of the V_j in each fiber span a dense subspace of ~~that~~ fiber.)

We can apply Gram-Schmidt to obtain an orthonormal basis. $\square$

This argument is basically due to Mingo & Phillips.

Fredholm Theory

We want to define a Fredholm operator on a continuous field.

Minimal requirement: the fibre operators should be Fredholm & the index should be locally constant.

In view of what we have seen, it is not clear ~~that~~ this is possible... but it <u>is</u>, as <u>Kasparov</u> discovered.

First, a warning example.
Let H_0 be the trivial field and let H_1 be the ~~range~~ kernel of the isometry V^* considered earlier.
This too is a continuous field.
Consider the projection
$$H_0 \longrightarrow H_1$$

whose adjoint exists and is the inclusion. Each fiber operator is Fredholm, but the fiberwise index is 0 at 0 and 1 elsewhere — the fiberwise index is not a locally constant function.

Moral: The concept of Fredholm family (that is, Fredholm operator on a continuous field) needs to be defined with a bit more care.

Here is how it is done — we shall use Atkinson's theorem that an operator on a single Hilbert space is Fredholm iff it is invertible modulo compact operators to guide us.

<u>Definition</u> An operator $K: H_0 \to H_1$ between continuous fields over a locally compact space X is <u>compact</u> if it is a norm-limit of operators of the form

$$s \longmapsto \sum_{i=1}^{n} \langle s_i, s \rangle s_i' ,$$

where s_i, s_i' are continuous sections that vanish at ∞ in the case where X is not compact; that is $\lim_{x \to \infty} \|s_i(x)\| = 0$ and $\lim_{x \to \infty} \|s_i'(x)\| = 0$.

<u>Definition</u> An operator between continuous fields is <u>Fredholm</u> if it is invertible modulo compact operators.

<u>Remark</u> If X is not compact, then the fiber operators F_x become <u>invertible</u> as $x \to \infty$. <u>All</u> the operators in the family have index zero.

The index of the fiber operators constituting a Fredholm operator is a locally constant function on X.

In fact:

<u>Theorem</u>* If X is compact, and if F is a Fredholm operator between continuous (countably generated) fields over X, then there is a compact perturbation F' of F for which $\ker F'$ and $\ker F'^*$ are vector bundles.

<u>Theorem</u> (Kasparov's version of Atiyah-Janich)
There is an isomorphism

$$K(X) \cong \text{homotopy classes of Fredholm families}$$

that assigns to a Fredholm family F $[\ker F] - [\ker F^*]$ if $\ker F$ & $\ker F^*$ are vector bundles.

These may be proved by the same methods used for Atiyah-Janich (minus Kuiper's theorem).

---

*To be honest, it may be necessary to add a trivial field to H_0 & H_1, after which we can assume all fields are trivial by the Dixmier-Douady theorem.