Kevin Costello - Topology in 2 dimensions & Frobenius Algebras

Note Title 9/5/2007

Surprisingly strong connection between homological algebra of moduli of surfaces & Frobenius algebras

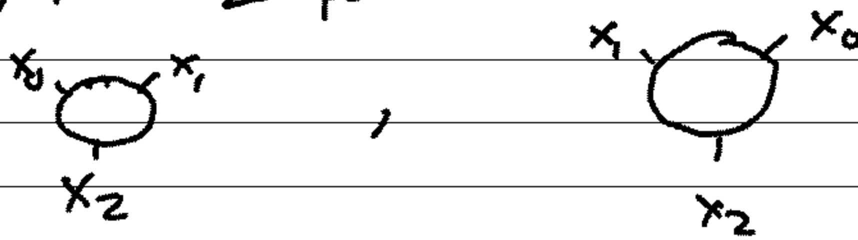
$D(n) = \{ n \text{ distinct points on } S^1 \} / PSL_2\mathbb{R}$

$S^1 = \partial D, \quad D \subseteq \mathbb{C}$

$= \{ n \text{ points on } \partial D \} / PSL_2\mathbb{R}$

$= \{ \text{discs with } n \text{ points on the boundary} \}$

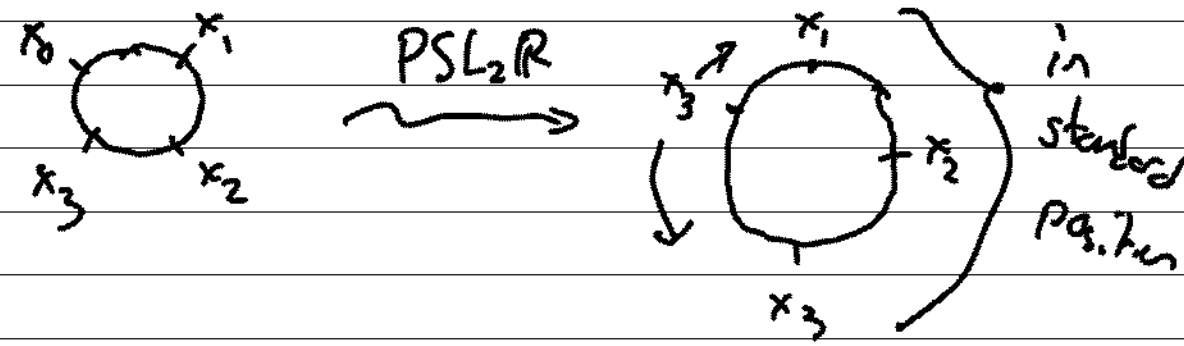
eg $D(3)$: 2 points:



$D(4)$: 6 connected components $\longleftrightarrow$ different orderings

S_4 action permutes these components

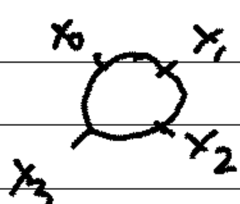
— suffices to describe one:



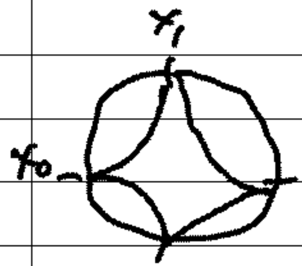
x_3 free to move from $-i$ to i

$\Rightarrow$ $D(4)$ is 6 copies of $(0,1)$

Let's compactify: want to replace by 6 copies of $[0,1]$ in a natural way.

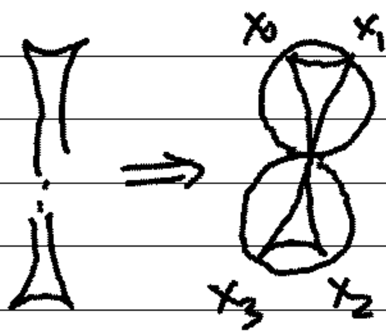


$\in D(4) \Rightarrow$ draw ideal polygon with vertices x_i (using hyperbolic geometry of disc)



If x_0, x_1 collide, polygon looks like (metrically)

In the lim $x_0 \longrightarrow x_1$, we find



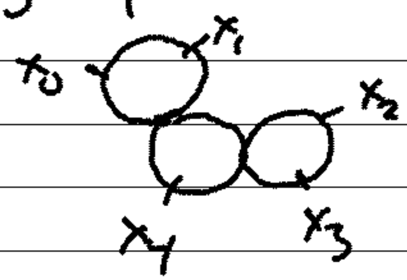
split into two triangles

$\rightarrow$ find two different discs glued together

$\longrightarrow$ add on singular discs

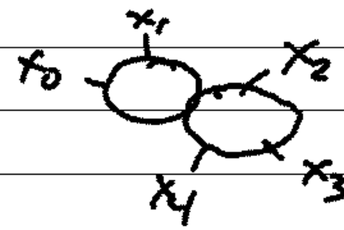
(picture of Fukaya: real algebraic version of Deligne-Mumford space)

Let $\overline{D}(n)$ be the compactification, allowing singular discs e.g. point in $\overline{D}(5)$:



lies in deepest stratum

--- limit of



as $x_2 \to x_3$

Once a disc has three special points, no moduli left — can't collapse further.

Discovery (Fukaya) : these spaces are the Stasheff polytopes, controlling homotopy associative algebras.

Let C_* be a nice chain complex functor:

$$C_r(\overline{D}(n)) = \left\{ \begin{array}{l} r \text{ dimensional} \\ \text{nice subspaces of } \overline{D}(n) \end{array} \right\}$$

(vector space span)

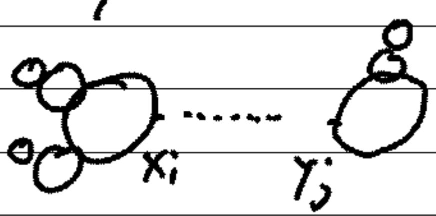
There are gluing maps between the $\overline{D}(n)$:

If $1 \le i \le n$, $1 \le j \le m$

$\Rightarrow$ a gluing map

$$\overline{D}(n) \; {}_i\!\!-\!\!-_j \; \overline{D}(m) \longrightarrow \overline{D}(n+m-2)$$

Pictorially:



glue x_i & y_j together to get a new node

What if we had a vector space with operations labelled by $\overline{D}(n)$ & compositions given by these gluing maps!

Let V be a chain complex with a non-degenerate symmetric inner product

For $1 \le i \le n$ $1 \le j \le m$ we have maps

$$\mathrm{Hom}(V^{\otimes n}, Q) \ {}_i\!\!-\!\!{}_j \ \mathrm{Hom}(V^{\otimes m}, Q)$$

$$\longrightarrow \mathrm{Hom}(V^{\otimes n+m-2}, Q)$$

$$(\varphi \ {}_i\!\!-\!\!{}_j \ \psi)(v_1, \dots, v_{n+m-1})$$

$$= \sum \varphi(v_1, \dots v_{i-1}; v', \dots v_{n-1}) \, \psi(v_n, \dots v'', \dots v_{n+m-2})$$

(ie trace over i,j factor)

where $\sum v' \otimes v'' \in V \otimes V$ is the inverse to $\langle , \rangle$

<u>Def</u> A $\bar{D}$-algebra is a chain complex V with a nondegenerate inner product & a map

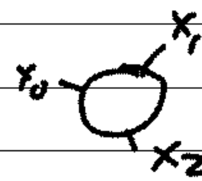
$$\Psi \quad C_*(\bar{D}(n)) \longrightarrow \mathrm{Hom}(V^{\otimes n}, Q)$$

which is an S_n-equivariant chain map

such that $\Psi(\alpha \circ_i \beta) = \Psi(\alpha) \circ_i \Psi(\beta)$

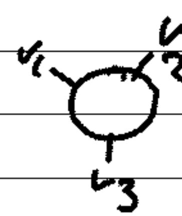
<u>Theorem</u> (Fukaya)

A $\overline{D}$-algebra is an A_∞-algebra with a "Frobenius" condition

<u>Proof</u> In $\overline{D}(3)$ have 

$\leadsto$ a map $V^{\otimes 3} \longrightarrow \mathbb{Q}$

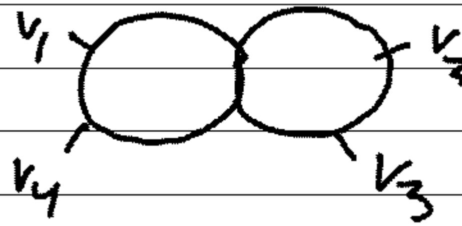
$\longleftrightarrow$ (using pairings) a map $m_2 : V^{\otimes 2} \longrightarrow V$

ie $\langle m_2(v_1, v_2), v_3 \rangle$ corresponds to 

The other point in $\overline{D}(3)$ gives the same m_2 up to permutation, by S_n-equivariance.

Look at $m_2(v_1, m_2(v_2,v_3))$: this is a map $V^{\otimes 4} \longrightarrow \mathbb{Q}$

corresponding to

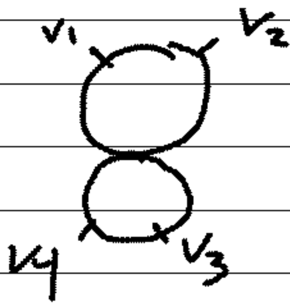


Is this product m_2 associative?

Compare with

$\langle m_2(m_2(v_1,v_2),v_3), v_4 \rangle$

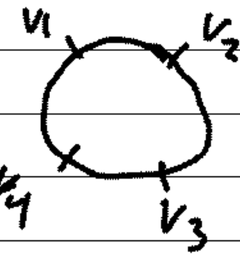
i.e.



: not equal $\Longrightarrow$ not associative!

But these two are connected by a natural one parameter family of discs:

Consider the one-dimensional chain in $\overline{D}(4)$ (in fact a connected component of $\overline{D}(4)$) of things like



– call it C_4.

Two boundaries:

$dC_4 =$  $-$  .

So our product is associative up to a particular homotopy.

Define $m_3 : V^{\otimes 3} \longrightarrow V$ so that

$$\langle m_3(v_1, v_2, v_3), v_4 \rangle : V^{\otimes 4} \longrightarrow \mathbb{Q}$$

corresponds to the chain C_4.

$$\Rightarrow \; dm_3(v_1, v_2, v_3) \pm m_3(dv_1, v_2, v_3) \cdots \pm m_3(v_1, v_2, dv_3)$$
$$= m_2(m_2(v_1, v_2), v_3) - m_2(v_1, m_2(v_2, v_3))$$

Likewise chains in higher moduli spaces give further homotopies, e.g. between different ways of inserting m_2 in m_3.

Frobenius axiom:

$\langle m_2(v_1, v_2), v_3 \rangle$ is cyclically symmetric

$\langle m_3(v_1, v_2, v_3), v_4 \rangle$ is cyclically antisymmetric

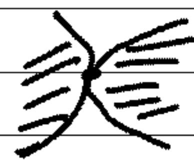
& so on.

Same picture holds for surfaces of all genus:

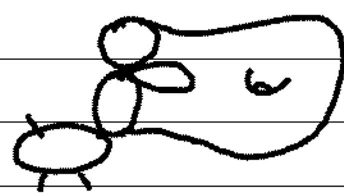
$$N_{\chi,k} = \left\{ \begin{array}{l} \text{connected Riemann surfaces} \\ \text{with boundary with} \\ k \text{ marked points on the boundary} \end{array} \right\}$$

Thus $N_{1,k} = D(k)$ (χ = Euler characteristic)

Partial compactification:
allow singular surfaces (ie <u>nodes</u> on the boundary, like the singularities we allowed before)



Call this space $\overline{N}_{\chi,k}$

eg  $\in \overline{N}_{-3,4}$

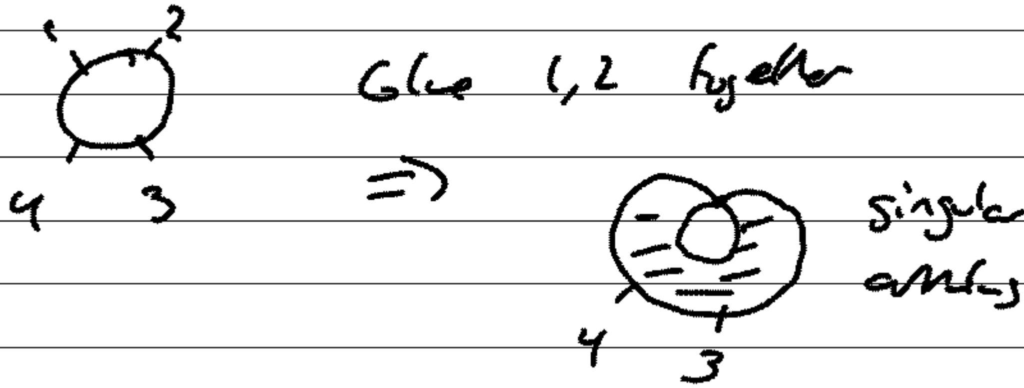
What algebraic structure do these encode?

$$\overline{N}_{\chi,n} \;{}_i\!\frown\!{}_j\; \overline{N}_{\chi',m} \longrightarrow \overline{N}_{\chi+\chi'-1,\,n+m-2}$$

But can also if $1 \le i < j \le n$
can glue i to j

$$\overset{i}{\underset{j}{C}}\ \overline{N}_{\chi,k} \longrightarrow \overline{N}_{\chi-1,\,k-2}$$

e.g.



<u>Def</u> An $\bar{N}$-algebra is a chain complex V with a nondegenerate inner product & maps $C_*(\bar{N}_{\chi,1}) \to \mathrm{Hom}(V^{\otimes n}, \mathbb{Q})$ respecting all the structures on both sides.

In particular an $\bar{N}$-algebra is a $\bar{D}$-algebra

<u>Theorem</u> Homotopically $\bar{N}$-algebras are the same as $\bar{D}$-algebras, ie homotopy associative algebras with Frobenius condition.

.... ie $\bar{N}$ can be built up from $\bar{D}$ by gluing

Topological formulation:

Let $\overline{D}_{\chi,n} \subset \overline{N}_{\chi,n}$ be the subset of surfaces built up from discs (ie irreducible components are all discs)

Then this inclusion is a homotopy equivalence.

Proof We'll give a deformation retraction

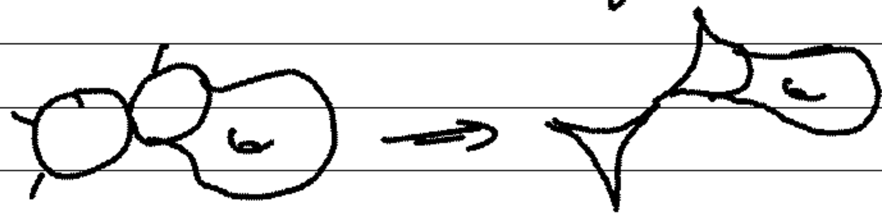
$$F : \overline{N}_{\chi,n} \times [0,1] \longrightarrow \overline{N}_{\chi,n}$$

$$F(\Sigma,0) = \Sigma \; , \; F(\Sigma,1) \in \overline{D}_{\chi,n}$$

$$\& \; F(\Sigma,t) = \Sigma \text{ if } \Sigma \in \overline{D}_{\chi,n} .$$

Take Σ & remove all marked points & nodes ($\Sigma \in \overline{N}_{\chi,n}$). This has a canonical (unique) hyperbolic metric with geodesic boundary

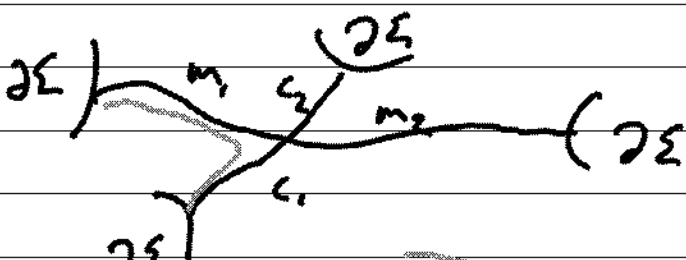
(+ completeness requirement)

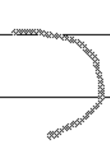


Consider geodesics on Σ° which start & end on $\partial \Sigma^\circ$ & are homotopically non-trivial in Σ (eg  not allowed)

Lemma: Any 2 such geodesics of shortest length are disjoint.

Suppose not: can join up to a shorter broken geodesic & then smooth:

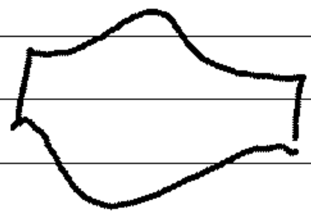
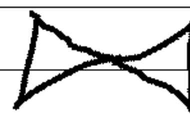


 has length $\leq c_1 + c_2$

$\Rightarrow$ smooth it to have strictly shorter length

$\Rightarrow$ Cut Σ along all shortest geodesics

To move Σ to the boundary,
insert at each cut a family R_t
of surfaces where $R_0 = 1$ does nothing

$R_t =$  $R_1 =$ 

turn our cut into a node
$\rightsquigarrow$ deformation retract onto a boundary

$\Rightarrow$ proceed by induction:
repeat this process until only discs
are left.