The Fundamental Lemma

Note Title 1/29/2010

* A conjecture about conjugacy classes in p-adic groups (Langlands - Shelstad) resolved by Ngô. Why do we care?

The Trace Formula

$A = \begin{pmatrix} a_{11} & & a_{ij} \\ & \ddots & \\ & & \ddots \end{pmatrix}$ matrix, eigenvalues λ_i

$\mathrm{Tr}\, A = \boxed{\sum a_{ii} = \sum \lambda_i}$

Geometric — direct, concrete; Spectral — meaningful

G finite, $\rho: G \circlearrowright V \simeq \mathbb{C}^n$ representation

Character $\chi(g) = \mathrm{Tr}\, \rho(g)$ is a class function - can write in two bases

$\boxed{\text{Conjugacy classes} \longleftrightarrow \text{irreducible characters}}$

Geometric — Spectral

LHS explicit & accessible typically, RHS is where content lies

eg $V = \mathrm{Fun}(\Gamma \backslash G) \circlearrowleft \mathbb{C}G$

Group algebra: $f = \sum f_i \delta_{g_i}$ combo of group elts

f acts on V with "matrix" (kernel function)

$K(x,y) = \sum_{\gamma \in \Gamma} f(x^{-1} \gamma y) \qquad x, y \in \Gamma \backslash G$

$\Rightarrow$ calculate trace geometrically as "$\int$ over $x = y$"

Frobenius Formula

$$\boxed{\sum_{\gamma\in\Gamma} a_\gamma \int_{[\gamma]\subset G} f = \sum_{\pi\subset V} m_\pi \operatorname{Tr}\pi(f)}$$

a_γ: $\mathrm{vol}\ \dfrac{Z_G(\gamma)}{Z_\Gamma(\gamma)}$ — explicit

$[\gamma]$: conjugacy class

m_π: multiplicity of irrep π in V — unknown

$\operatorname{Tr}\pi(f)$: character of irrep

Infinite gp example: $G = \mathbb{R}$, $\Gamma = \mathbb{Z}$

$\mathbb{Z}\backslash\mathbb{R} = S^1 \implies L^2(S^1)$ spanned by $e^{2\pi i n x}$

$f \in C_c^\infty(\mathbb{R})$ test function $\Rightarrow$

Poisson Summation:

$$\boxed{\sum_{n\in\mathbb{Z}} f(n) = \sum_{\lambda\in 2\pi i\mathbb{Z}} \hat{f}(\lambda)}$$

Tate's Thesis: apply Poisson summation with $\mathbb{R} \rightsquigarrow \mathbb{R}\times\prod_p \mathbb{Q}_p$ adeles & recover class field theory: RHS knows abelian part of $\mathrm{Gal}\,\overline{\mathbb{Q}}/\mathbb{Q}$.

Langlands program: vast nonabelian generalization. Asserts mysteries of universe captured by harmonic analysis on certain locally symmetric spaces.

Take G reductive group over $\mathbb{Q}$

e.g. matrix groups like $GL, SL, SO, Sp, \ldots$:
- can plug in $\mathbb{Q}, \mathbb{R}, \mathbb{Q}_p, \ldots$ as coefficients of matrices.

$\rightsquigarrow$ assign a space X_G with $G(\mathbb{R}), G(\mathbb{Q}_p)$ $\forall p$ as symmetries

$G = SL_2$: $X = SL_2\mathbb{Z} \backslash \mathbb{H}$

home of modular forms

$SL_2\mathbb{R}$ symmetry evident from $\mathbb{H} = SL_2\mathbb{R}/SO_2$

$SL_2\mathbb{Q}_p$ symmetry better hidden: captured by Hecke operators

.... in fact $X = SL_2\mathbb{Q} \backslash {\prod_{v=p,\mathbb{R}}}' SL_2\mathbb{Q}_v \,/\, \underline{K}$ max compact

Automorphic representations [generalize modular forms]
- spectrum of $L^2(X)$

... universal nuggets of truth, control Galois group of $\mathbb{Q}$ (& number fields) via reciprocity laws..

Access: via Arthur-Selberg trace formula
- version of Frobenius/Poisson for complicated noncompact spaces X_G: have discrete & continuous spectrum, operators not trace class, integrals diverge, ... monumental achievement!

Spirit: (criminally simplified) also have continuous part, ...

$$\sum_{\gamma \in G(\mathbb{Q})} a_\gamma \underbrace{\int_{[\gamma]} f}_{\text{product of orbital integrals for all } G(\mathbb{Q}_p)} \sim \sum_{\pi \text{ automorphic}} m_\pi \operatorname{Tr} \pi(f)$$

Can't stare at directly or get blinded..
automorphic reps are like precious stones, have to hoard slowly & carry along with you from place to place:
transfer automorphic reps between different groups, by comparing LHS of trace formulas for different G ....
— key behind great modern successes, e.g. Shimura-Taniyama Conjecture ($\Rightarrow$ FLT) (capture elliptic curves via modular forms)

Fundamental Question:

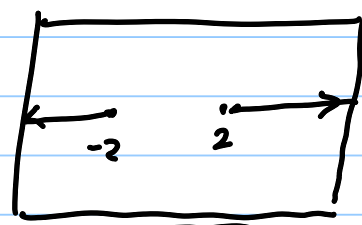
How do you compare conjugacy classes (semisimple) (hence orbital integrals & geometric sides) for <u>different</u> <u>groups</u>?

For matrices: can compare eigenvalues or better coefficients of characteristic polynomial ... said more geometrically, zero locus of char. poly finite subset of the line (spectrum of matrix).

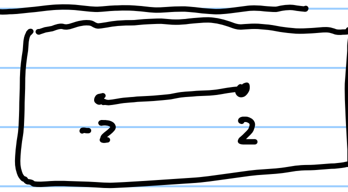
[In general: values of invariant polynomials]

eg $SL_2\mathbb{R}$: semisimple elts look like

$\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$ Trace:



$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ Trace



<u>Problem</u>: R_θ & $R_{-\theta}$ have <u>same</u> char polynomial $\Longleftrightarrow$ conjugate in $SL_2\mathbb{C}$ (ie over algebraic closure) but NOT conjugate in $SL_2\mathbb{R}$!

--- <u>stable</u> <u>conjugacy</u>. (doesn't occur for GL_n)

To make trace formula really usable need to have a version depending on <u>stable</u> conjugacy classes ⇒ can compare for different groups & not lose precious gems.

<u>Fundamental Lemma</u> (Langlands-Shelstad)

Orbital integrals can be decomposed into combos of integrals over stable conjugacy classes in <u>endoscopic groups</u> of G:

a collection of combinatorically related groups — share tori, Weyl group information, but no actual homomorphisms between them

– e.g. SO_{2n} is an endoscopic group of Sp_{2n}....

Difficulty: need to compare integrals on only loosely related groups!

Langlands, Shelstad, Kottwitz, Waldspurger, Arthur: reduce "stabilization" of trace formula (& much else) to FL

– eg allows one to classify reps of all classical groups in terms of GL_n ... used in proof of Sato-Tate conjecture etc etc

Magic: F.L. for p-adic groups follows from F.L. for $G(\mathbb{F}_p((t)))$ almost all p
— theorem in logic (Hales-Cluckers-Loeser) or brute force (Waldspurger)

Reinterpretation (Kazhdan-Lusztig-Bernstein)
Use Weil conjectures (Deligne) to interpret orbital integrals as Euler characteristics of some varieties over $\mathbb{F}_p$ — now
$\Rightarrow$ geometric study by Goresky-Kottwitz-MacPherson, Laumon, resolves some cases.
— at this point might as well replace $\mathbb{F}_p$ by $\mathbb{C}$, study Euler characteristics of singular complex varieties

Ngô: 1. Globalization
2. New underlying symmetries
3. Categorification

1. Analogy between number fields & function fields:

$$\mathbb{Q} \leftarrow\!\dashrightarrow \mathbb{F}_p(Y) \leftarrow\!\dashrightarrow \mathbb{C}(\Sigma)$$

rational functions on

curve/$\mathbb{F}_p$ Riemann surface

completing $\mathbb{Q}_p \leftarrow\!\dashrightarrow \mathbb{F}_p((t)) \leftarrow\!\dashrightarrow \mathbb{C}((t))$

all look about same

Analog of locally symmetric spaces X_G:

$Bun_G \Sigma$ = moduli of G-bundles on Σ

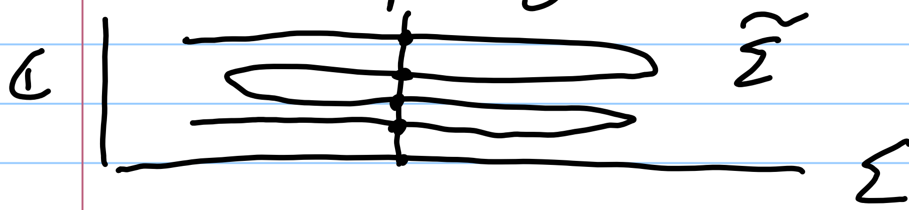
GL_n: $Bun_n \Sigma$ = " " rank n vector bundles on Σ

- very complicated space.

Ngô discovered that the F.L. has a natural interpretation in terms of the <u>Hitchin integrable system</u>:

$$\begin{tikzcd}
\underset{(\eta=0)}{Bun_n \Sigma} \arrow[r, phantom, "\subset"] \arrow[d] & \mathcal{M}_n \Sigma = \left\{ \begin{array}{l} (V,\eta)\ V \text{ rk } n \text{ bundle} \\ \eta \text{ endomorphism of } V \end{array} \right\} \arrow[d, "\text{Hitchin}"] \\
\{\Sigma \times n\{0\}\} \arrow[r, phantom, "\in"] & \left\{ \begin{array}{l} n\text{-fold branched covers} \\ \text{of } \Sigma \text{ inside } \Sigma \times \mathbb{C} \end{array} \right\}
\end{tikzcd}$$

Hitchin map: graph spectrum of η in $\Sigma \times \mathbb{C}$



generically nice smooth surface.
Graph also eigenspaces $\Rightarrow$ line bundle on $\tilde{\Sigma}$, together recover (V, η)

- ie generic fibers are Jacobians

$\mathrm{Jac}\,\tilde{\Sigma} = $ (deg 0) line bundles on $\tilde{\Sigma} \simeq (S^1)^{2\tilde{g}}$

Hitchin's integrable system: flows are straight line flows on tori.

* Ngô discovered an interpretation of FL in terms of Euler characteristics of (singular) fibers of Hitchin's map!

Endoscopic groups aren't subgroups, but associated spectral curves determined by torus, Weyl group data $\longrightarrow$ can map Hitchin systems for them into that for G..

* Global FL picture implies local one: ..... all contributions come from branch points over $\Sigma \longleftrightarrow$ local data.

2. Symmetries: Ngô "integrated" Hitchin's flows through a fundamental observation:

For an $n \times n$ matrix A we can construct a family of commuting matrices via "functional calculus": f polynomial on $\mathbb{A}^1$, can make sense of $f(A)$

Since A satisfies char poly(A) $\Rightarrow$ only get contributions from functions on the spectrum of A = zeros of char(A).

$\Rightarrow$ n-dim commutative group (from invertible functions) centralizing A.

Ngo generalized this to all reductive G:
rank(G)-dim commutative subgroup of centralizer of any conjugacy class.

Totally elementary but new!

Now replace $A \rightsquigarrow \eta$: get natural group of symmetries of Hitchin fibers, "integrating Hitchin flows".

3. Finally: FL is a decomposition of a number as a sum... but numbers don't have symmetries!

Categorify: [Pass to geometric Langlands setting]

Eulr chars $\rightsquigarrow$ cohomology
numbers $\rightsquigarrow$ vector spaces
functions $\rightsquigarrow$ bundles, or sheaves

H^*(fiber) $\circlearrowleft$ Ngô's group of symmetries — can break into irreducible summands! Ngô shows these summands are precisely the categorified endoscopic orbital integrals — appear naturally from geometry of spectral curves $\tilde{\Sigma}$.... involves algebro-geometric study of degenerating families of tori.

.... H^*(fiber) vary nicely (as perverse sheaves) over base, can describe using deformations to smoother fibers.
[key advantage of categorifying fns $\rightsquigarrow$ sheaves & using "tameness" of global algebraic geometry]

$$
\begin{tikzcd}
 & \mathrm{Gal}\,\bar{\mathbb{Q}}/\mathbb{Q} \arrow[d, "\text{Langlands}"] & & \text{Gauge theory} \arrow[d, "\text{TFT}"] \\
\text{Global} & \text{Automorphic forms} \arrow[d, "\text{trace formula}"] & & \text{Hitchin fibers} \\
\text{Local} & \text{orbital integrals} \arrow[rr, "\text{magic}"'] & & \text{affine Springer fibers} \arrow[u, "\text{Ngo}"] \\
 & \text{Number Fields} & & \text{Function Fields}
\end{tikzcd}
$$