Note Title 9/22/2008

Catharina Stroppel: Crystal bases, Hecke algebras & equivalences of categories

Warmup: representations of

$$\mathfrak{sl}_2\mathbb{C} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a + d = 0 \right\}$$

basis $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$[h,e] = 2e$, $[h,f] = -2f$

& $[e,f] = h$

Irred representations (finite dimensional):

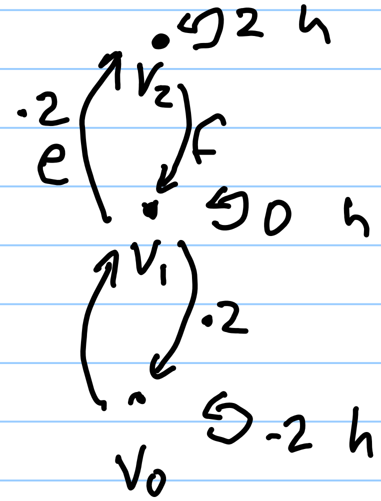
irreps $\longleftrightarrow \mathbb{N}$

$V \longmapsto \dim V$

denote $V(n)$ the irrep of dimension $n+1$.

Explicitly: basis $v_0, v_1, \ldots v_n$

$n=2$



Now categorify these representations!

a. Very weak sense:

Find a category $\mathcal{C}$ (k-linear, abelian or triangulated)
together with an isomorphism of vector spaces $[\mathcal{C}] := \mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathcal{C}) \xrightarrow[\Phi]{\sim} V$

$K_0(\mathcal{C})$: free abelian group on isomorphism classes of objects in $\mathcal{C}$
$[M]$ $M \in \mathcal{C}$ / relation:
$[C] = [A] + [B]$ if there is a short exact sequence $A \hookrightarrow C \twoheadrightarrow B$
(or distinguished triangle)

Example $\mathcal{C} = \mathbb{C}$-mod = fin dim vector spaces. $\mathbb{C} \hookrightarrow \mathbb{C}^2 \twoheadrightarrow \mathbb{C}$

$\Rightarrow$ $[\mathbb{C}^2] = 2[\mathbb{C}]$ $\rightsquigarrow$
$K_0 = \mathbb{Z}$.

$\mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathcal{C}) \simeq \mathbb{C}$: categorify 1-dim vector space!

Now should also have exact functors
$E, F, H : \mathcal{C} \circlearrowleft$
so that the induced maps $[E], [F], [H]$
on $[\mathcal{C}]$ satisfy the relations of $\mathfrak{sl}_2 \mathbb{C}$.

Example $V = V(2)$

Take $\mathcal{C} = \mathcal{C}_{-2} \oplus \mathcal{C}_0 \oplus \mathcal{C}_2$

with each $\mathcal{C}_i \simeq \mathbb{C}\text{-mod}$.

Basis $[\mathbb{C}]$ in each component.

$E|_{\mathcal{C}_{-2}} : M \in \mathcal{C}_{-2} \longrightarrow M \in \mathcal{C}_0$

$E|_{\mathcal{C}_0} : M \in \mathcal{C}_{-2} \longrightarrow M \oplus M \in \mathcal{C}_2$

F similarly,

$H = 2\,\mathrm{id} \oplus 0 \oplus 2\,\mathrm{id}$

We'll change this to a more interesting categorification:

$\mathcal{C} = \mathbb{C}\text{-mod} \oplus \mathbb{C}[x]/x^2\text{-mod} \oplus \mathbb{C}\text{-mod}$

Basis $[\mathbb{C}]$ $[\mathbb{C}[x]/x^2]$ $[\mathbb{C}]$

$$E|_{e_{-2}} = \mathrm{Ind}_{\mathbb{C}}^{\mathbb{C}[x]/x^2} - = \mathbb{C}[x]/x^2 \otimes_{\mathbb{C}} -$$

Note our basis is <u>not</u> a $\mathbb{Z}$-basis, only a $\mathbb{C}$-basis! [ring has infinite homological dimension....]

$E|_{e_0} = \mathrm{Res}^{\mathbb{C}}_{\mathbb{C}[x]/x^2}$, F similarly

So we get two different bases for the complex representation ring from our categorification!

using simple module $\mathbb{C}$ gives "dual canonical basis", using projective module $\mathbb{C}[x]/x^2$ gives "crystal (or canonical) basis"

<u>Remark</u> For $V(n)$:

take $e_n = \bigoplus_{i=0}^{n} H^*(Gr(i,n))$-mod

- $\mathbb{C}[x]/x^2$ is the cohomology of $\mathbb{P}^1$

$$H^*(Gr(i,n)) \longleftarrow H^*\left(\begin{matrix}\text{two-step flags in } \mathbb{C}^n \\ \text{of dim } i, i+1\end{matrix}\right) \longrightarrow H^*(Gr(i+1,n))$$

give our induction & restriction functors

Let's change the definition of categorification to see if we can make it <u>unique</u>.

b. Categorification, weak sense (Chuang & Rouquier):

- $\mathcal{C}$ is categorification in very weak sense
- $\mathcal{C}$ artinian, noetherian, k-linear, abelian
- endomorphism rings of simple objects are just k.
- (E, F) adjoint pair of exact functors
- simple objects are weight vectors for H

(ie H rescales classes of simple objects)

- F is isomorphic to a <u>left</u> adjoint of E.

<u>Proposition</u> [CR] If $\mathcal{C}$ is a weak categorification of V, & for $V_\lambda \subset V$ weight space denote $\mathcal{C}_\lambda \subset \mathcal{C}$ full subcategory generated by objects with class in V_λ $\Longrightarrow$

$$\mathcal{C} = \bigoplus \mathcal{C}_\lambda .$$

C. Categorification (in strong sense):

- weak categorification
- $T \in End(E^2)$
  $X \in End(E)$ satisfying certain affine Hecke relations

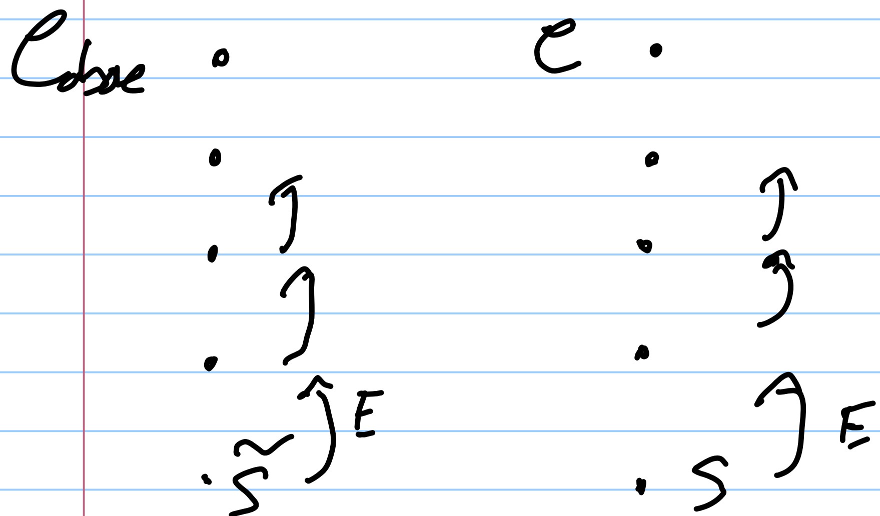
<u>Theorem</u> [CR] • Every finite dim sl_2 module has a (strong) categorification $\mathcal{C}_{above}$

- let $\mathcal{C}$ be another categorification (of the same V) & $M \in \mathcal{C}$ simple with $F \cdot M = 0$

If M is projective $\Rightarrow$ $\mathcal{C} \simeq \mathcal{C}_{above}$!

— gives tool to show equivalences of categories without knowing any functors between them.

Idea of proof (speaker's POV):



both simples & projectives $\Rightarrow$ generate equivalent categories

Now apply E to these equivalent bottoms $\longrightarrow$ construct lots of projective objects — in fact get a projective generator. Operators T, X control their endomorphism rings $\Rightarrow$ the categories are equivalent.

- this is a categorification of an identification of two crystal bases.

- apply this to understand f.d. modules over super Lie algebras

<u>Application</u> Knot theory.

<u>Problem</u> Categorify tensor products
$V(\lambda_1) \otimes \ldots \otimes V(\lambda_r)$
- or rather their quantized versions:

$sl_2 \rightsquigarrow U_q sl_2$:
algebra over an indeterminate q
(ie $\mathbb{C}(q)$-algebra) generated by
$E, F, K^{\pm 1}$. Relations:

$$KE = q^2 EK, \quad KF = q^{-2} FK$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

Irreps look the same, only use square brackets around numbers:

$e v_i = [i+1]\, v_{i+1}$

$f v_i = [n-i+1]\, v_{i-1}$

$h v_i = [2i-n]\, v_i$ etc

where the q-numbers $[k] = \dfrac{q^k - q^{-k}}{q - q^{-1}}$

$= q^{k-1} + q^{k-3} + \cdots + q^{1-k}$

Special case $\underbrace{V(1) \otimes \cdots\cdots \otimes V(1)}_{n \text{ times}}$ $\circlearrowleft \mathbb{C}_q[S_n]/_{I}$

as module for $U_q sl_2$

$\mathbb{C}_q[S_n]$ - deformed version of group algebra of S_n, acting mod-ulo some ideal $\rightsquigarrow$

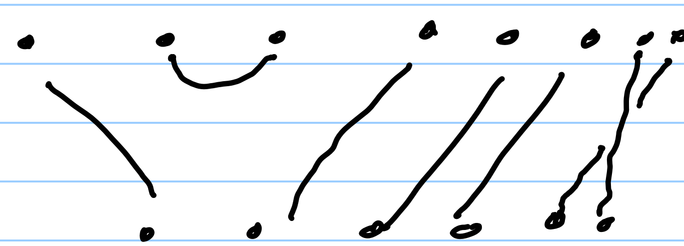
get Temperley-Lieb algebra $:=$

algebra with basis given by crossingless (n,n) tangles

& multiplication = concatenation

with circles $= q + q^{-1}$

e.g. $(8,6)$ tangle:



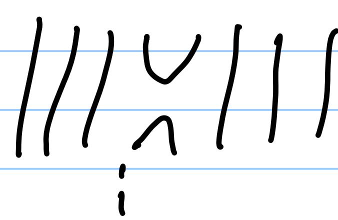
Crossingless: no 

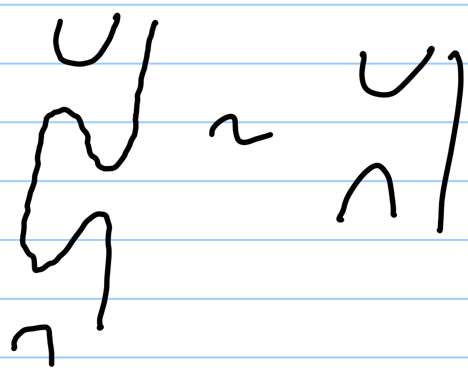
Temperley-Lieb algebra has generators c_i $\quad 1 \le i \le n-1$ $\quad$ [$c_i \sim 1+$ simple transpositions]

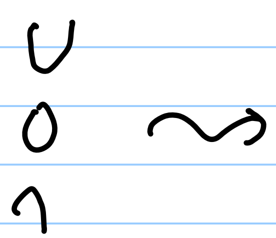
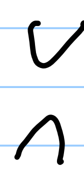
relations $c_i c_j = c_j c_i$ if $|i-j|>1$

$c_i c_j c_i = c_i \quad |i-j|=1$

$c_i^{\,2} = (q+q^{-1})\, c_i$

$c_i =$ 

$c_1 c_2 c_1$  c_1

$c_1^{\,2}$  $\rightsquigarrow$ $(q+q^{-1})$ 

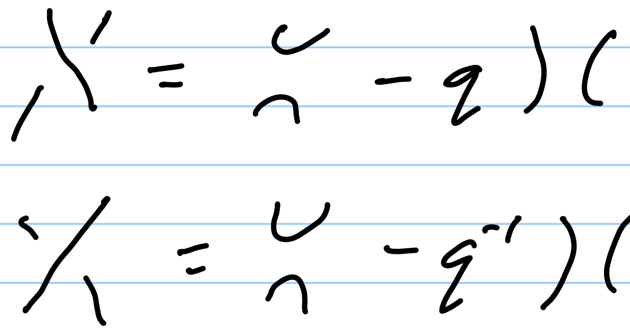
Jones, Reshetikhin–Turaev defined knot/tangle invariants as follows:

1. To (n,n) tangle with no crossings

   $\Rightarrow$ get element of TL algebra

   $\Rightarrow$ endomorphism of $V_q(1)^{\otimes n}$

2. Arbitrary (n,n) tangle: resolve by



$\Rightarrow$ get crossingless matchings

$\rightarrow$ get an endo of $V_q(1)^{\otimes n}$

3. Extend to arbitrary tangles

Remark: For $(0,0)$ tangles – ie knots! – get a map $\mathbb{C} \longrightarrow \mathbb{C}$ depending on q

$\varphi(1)$ = Jones polynomials.

Why categorification:
lift all of the above to categorified level, find more refined invariant of knots which distinguishes knots that Jones doesn't

Theorem (Stroppel; build on Bernstein-Frenkel-Khovanov)

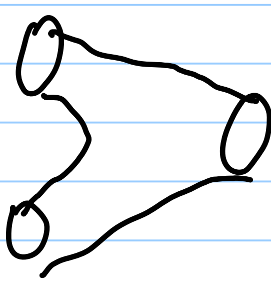
1. There is a categorification of $V_q(1)^{\otimes n}$ for n, $\mathcal{C}(n)$

2. There is a functor
$$\{\text{tangles}\} \longrightarrow \text{category } CAT$$

- objects $\mathbb{N} \ni n \longmapsto D^b(\mathcal{C}(n))$

1- morphisms tangles $\longmapsto$ functors

2- morphisms cobordisms $\longmapsto$ natural transformations



— recover Khovanov's homology from here