Representation theory and a strange duality for symplectic varieties

Ben Webster
(joint with Tom Braden, Tony Licata, and Nick Proudfoot)

MIT/Oregon

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Outline

References:

This slide show can be downloaded from

`http://math.mit.edu/~bwebster/austin-dual.pdf`

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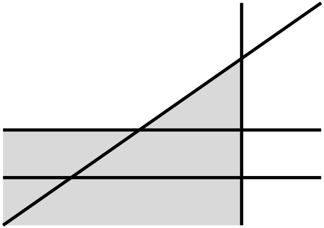

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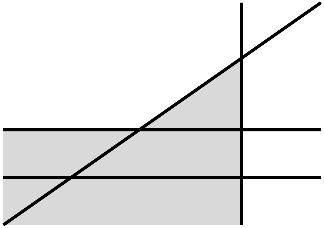

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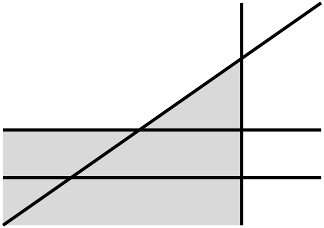

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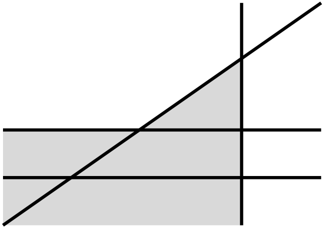

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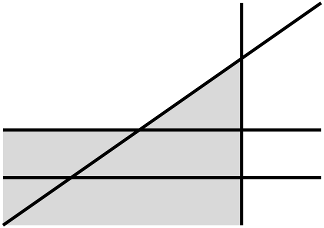

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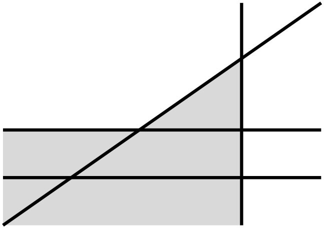

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This picture above is of $V + \xi$. The hyperplanes in the arrangement are the vanishing sets of $t_i|_{V+\xi}$ (where the t_i are the coordinates on $\mathbb{R}^n$).

The **chambers** of $\mathcal{V}$ are the connected components of $(V + \xi) \cap (\mathbb{R}^\times)^n$.

We call a chamber **bounded** if ν achieves a maximum on it. We let $\mathcal{B}$ denote the set of bounded chambers.

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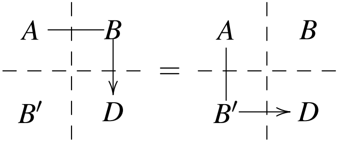

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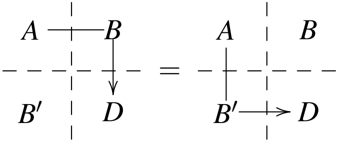

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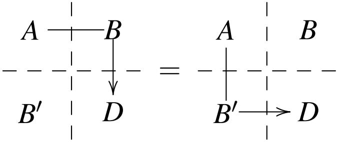

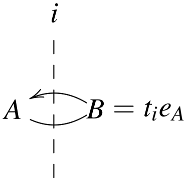

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- *$A(\mathcal{V})$ is quasi-hereditary.*
- *$A(\mathcal{V})$ is Koszul.*
- *The center $Z(A(\mathcal{V}))$ is the reduced Stanley-Reisner ring of $\mathbb{R}^n \to V^*$.*

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Note: these are Koszul dual!

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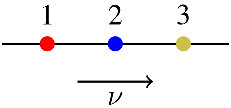

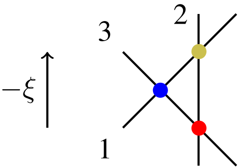

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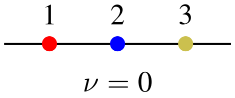

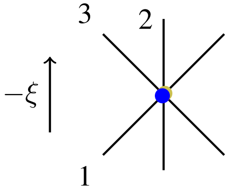

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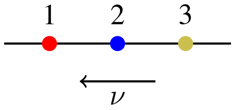

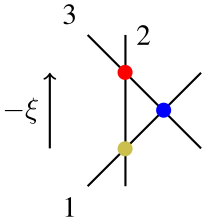

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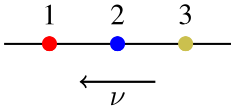

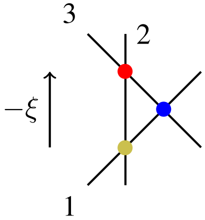

Theorem (BLPW)

$(A(\mathcal{V}))^\star \cong A(\mathcal{V}^\vee)$

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These isomorphisms are not canonical at all. In fact, they seem to only be unique up to an action of $\pi_1(\mathsf{Pol}_{\mathbb{C}}(V))$, the complexification of the spaces of choices of generic polarization of V.

$\mathsf{Pol}_{\mathbb{C}}(V)$ is the complment of a new hyperplane arrangement called the **doubled secondary arrangement**.

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- If you're an algebraist: an A-brane is a representation of a deformation quantization of functions on X.
- If you're a symplectic geometer: an A-brane is an object in the Fukaya category of X. (Just for motivation!)

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Algebraic symplectic implies Calabi-Yau, so it is very restrictive.

We'll be interested in a smooth symplectic variety $\tilde{X}$ which is a resolution of an affine cone X (i.e. X is an affine variety invariant under scaling). In this case, we say $\tilde{X}$ is a **symplectic resolution**.

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The universal enveloping algebra of $\mathfrak{g}$ is a deformation quantization of $\mathcal{N}$, so the BGG category $\mathcal{O}$ obviously fits into the algebraic definition of A-branes I gave. For the geometric one, this is trickier, but a theorem:

Theorem (Beilinson-Bernstein, Nadler-Zaslow)

There is in an inclusion $(\mathcal{O}_{\mathfrak{g}})_0 \hookrightarrow \mathrm{Fuk}(T^*G/B)$.

Hypertoric varieties

What symplectic cone corresponds to a hyperplane arrangement? If V is defined over $\mathbb{Z}$, then $\mathbb{C} \otimes_{\mathbb{R}} V^{\perp}$ is the Lie algebra of a subtorus $T \subset (\mathbb{C}^*)^n$. As always, we have a canonical moment map $\mu : T^*\mathbb{C}^n \to \mathfrak{t}^*$.

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$$\mathfrak{M}_\alpha = \mu^{-1}(0)/\!\!/_\alpha T = \bigsqcup_{v \in V} N^*(T \cdot v)/\!\!/_\alpha T$$

and obtain a **hypertoric variety**, closely tied to the combinatorics of T acting on V. For $\alpha = 0$ this is an cone, and for α generic, a symplectic resolution of $\mathfrak{M}_0$. (You might prefer to think of this as a hyperkähler reduction).

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$$\mathfrak{M}_{\alpha} = \mu^{-1}(0) /\!\!/_{\alpha} T = \bigsqcup_{v \in V} N^*(T \cdot v) /\!\!/_{\alpha} T$$

and obtain a **hypertoric variety**, closely tied to the combinatorics of T acting on V. For $\alpha = 0$ this is an cone, and for α generic, a symplectic resolution of $\mathfrak{M}_0$. (You might prefer to think of this as a hyperkähler reduction).

You can think of this as an "enhanced cotangent bundle" to the toric variety $\mathbb{C}^n /\!\!/_{\alpha} T$.

Hypertoric varieties and hyperplane arrangements

Our original data can be recovered as the affine hyperplane arrangement $(\ker\iota, \alpha, -)$ where $\iota : \mathbb{R}^n \to \mathfrak{t}_{\mathbb{R}}^*$ is the natural map.

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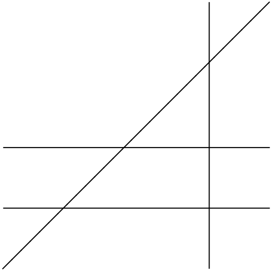

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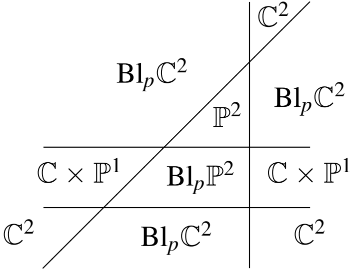

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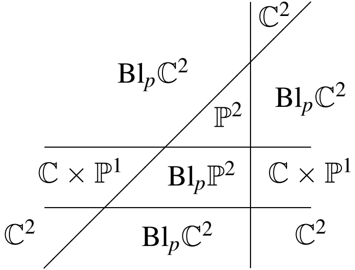

Proposition

Each toric variety X_C corresponding to a chamber C in the complement of the H_i's is a Lagrangian subvariety of $\mathfrak{M}_\alpha$, and $\mathfrak{M}_\alpha$ is a symplectic plumbing of their cotangent bundles.

$A(\mathcal{V})$ and geometry

The category $A(\mathcal{V}) - \mathsf{mod}$ for a polarized arrangement $\mathcal{V} = (V, \xi, \nu)$ has a geometric interpretation similar to that of $\mathcal{O}_{\mathfrak{g}}$.

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The algebra M_V can be constructed by non-commutative Hamiltonian reduction of the algebra of differential operators $\mathcal{D}_{\mathbb{C}^n}$ by T, and analyzed explicitly.

This deformation quantization of $\mathfrak{M}_V$ can be regarded as an analogue of the universal enveloping algebra, and one can search for analogues of all results of Lie theory. But that's another talk.

Deformation quantization

I believe that what I've told you thus far is just a hint of a much larger picture incorporating many more symplectic varieties. Let $\tilde{X}$ is a smooth symplectic variety which is a resolution of singularities of an affine cone X.

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Proposition (Bezrukavnikov-Kaledin)

There is a universal family A_X^λ of deformation quantizations of X over $H^2(\tilde{X})$.

Examples

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"Dragons" is a slight exaggeration; we know what the algebras are, but as far as I know, there is no literature on them.

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Based on various pieces of evidence, Braden, Licata, Proudfoot and I have suggested that this should reflect some kind of underlying duality $X \leftrightarrow X^\vee$ between symplectic cones. My collaborators originally dubbed this "symplectic duality" but it seems that for talking with physicists in the audience, the name should be shortened to "S-duality."

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Observation

Our examples coincide with a notion of duality in physics; they are the Higgs branches of mirror dual 3-dimensional gauge theories.

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Simplest interesting example: $T^*\mathbb{P}^{n-1} \Leftrightarrow \widetilde{\mathbb{C}^2/\mathbb{Z}_n}$ or, in terms of cones, $M^{\mathrm{rk}\,1}_{n\times n} \Leftrightarrow \mathbb{C}^2/\mathbb{Z}_n$.

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While the representation theoretic interpretation of duality is probably the most exciting, there are other, more geometric interpretations as well.

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The spaces $\tilde{X}$ and $\tilde{X}^\vee$ carry actions of tori S and T such that the fixed points $\tilde{X}^S$ and $(\tilde{X}^\vee)^T$ are in natural bijection.

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For physicists, this seems to be the fact that fixed points are "supersymmetric vacua."

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For physicists, these seem to come from the other branches of the moduli space of vacua.

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We also expect a kind of recursion (or perhaps you could call it functoriality)

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If the strata X_α and $X^\vee_{\alpha^\vee}$ match under this bijection, then X_α is S-dual to the **slice** *to $X^\vee_{\alpha^\vee}$ in $X^\vee$*

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- For hypertoric varieties, this is the compatibility of Gale duality with restriction and localization.
- For nilpotent matrices and spaces of A_n instantons, there is good evidence for this (some of it coming from physics).

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Intermission!

Flow-in cells

Let we now try to give you some slightly more precise geometry:

I'll assume for now that we have a torus T with a fixed choice of $\mathbb{C}^* \hookrightarrow T$, acting with isolated fixed points on $\tilde{X}$, which is a symplectic resolution of a cone $X \xleftarrow{\pi} \tilde{X}$.

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For each $x \in X^T$, I have the Lagrangian flow-in cell

$$F_x = \{y \in X | \lim_{t \to 0} t \cdot y = x\}.$$

Conjecture

If X_α is the smallest "special" stratum containing $\pi(F_x)$, then $X^\vee_{\alpha^\vee}$ is the smallest "special" stratum containing $\pi(F_{x^\vee})$

Flow-in cells

For a taste of what this says: the generic stratum and 0 stratum must switch under $\vee$, so this says $F_x \subset \pi^{-1}(0)$ iff $F_{x^\vee}$ isn't contained in any smaller stratum.

That is, duality switches core components (components of $\pi^{-1}(0)$) and MV cycles (components of the flow-in of the identity in $X^\vee$).

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A similar, but more complex phenomenon seems to hold for the intermediate strata: now we obtain an MV cycle in X_α and a core component of the resolution of the slice, which switch under duality.

This should be thought of as a generalization of the cell theory of category $\mathcal{O}$, which is what we obtain in the case of the flag variety.

Equivariant cohomology of $\tilde{X}$

Now, assume $\tilde{X}$ is equivariantly formal. That is, we have an injection

$$H_T^*(\tilde{X}) \to H_T^*(\tilde{X}^T) \cong \oplus_{a\in\tilde{X}^T} H_T(\{a\}).$$

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Let $R_{\tilde{X}}$ be the subring of $H_T^*(\tilde{X})$ generated by H_T^2 over H_T^0. More geometrically, we have

$$\operatorname{Spec} R_{\tilde{X}} = \bigcup_{a\in\tilde{X}^T} H_2^T(\{a\}) \subset H_2^T(\tilde{X}).$$

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There is an obvious “duality” on subspace arrangements, sending all subspaces to their annihilator. Let

$$R_{\tilde{X}}^\vee = \mathbb{C}[\bigcup_{a\in\tilde{X}^T} H_2^T(\{a\})^\perp] \subset H_2^T(\tilde{X})^*.$$

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Observation (Goresky-MacPherson, BLPW)

For all the examples above where the natural torus action has isolated fixed points, the "symplectic dual" $\tilde{X}^\vee$ also has an action of a torus S such that

$$R_{\tilde{X}}^\vee = R_{\tilde{X}^\vee}.$$

Examples

To keep up with our running examples:

Variety $\tilde{X}$	Spec R	**Duality**
T^*G/B	$\bigcup_{w\in W}\Gamma_w \subset \mathfrak{t}^* \oplus \mathfrak{t}^*$	Langlands
$\mathfrak{M}_{\mathcal{V}}$	$\bigcup_{\beta \text{ a basis of } V^*} \mathbb{C}^\beta \subset \mathbb{C}^n$	Gale
$\mathsf{Hilb}^n(\mathbb{C}^2)$	$\bigcup_{\lambda\vdash n}(1, \mathrm{Con}(\lambda)) \subset \mathbb{C}^2$	self-dual

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Localization duality for Koszul algebras

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Assume that A is quasi-hereditary, and the center $Z(\grave{A})$ is also flat.

Let R_A be the subalgebra of $Z(\grave{A})$ generated by $Z(\grave{A})_2$. As before, $\operatorname{Spec} R_A \subset Z(\grave{A}^\star)$ is a union of subspaces. Let $R_A^\vee$ be the coordinate ring of the union of the annihilators.

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As a corollary, proving a categorical duality would imply the cohomological duality on the previous page.

Geometric representation theory

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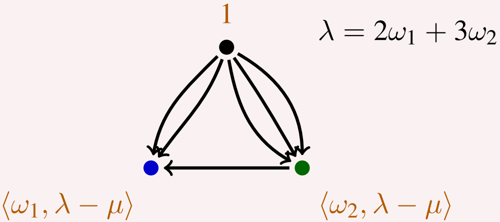
Each of these is worthy of a talk series in and of itself, but let me try to summarize the most important points.

Nakajima quiver varieties

Pick your favorite quiver (oriented graph), and let $\mathfrak{g}$ be the Kac-Moody algebra for that quiver.

Attached to a highest weight λ and weight space $\mu = \lambda - \sum_i d_i\alpha_i$, we have a Nakajima quiver variety $\tilde{\mathfrak{Q}}_\mu^\lambda$. This is the moduli space of stable representations of the preprojective algebra for a quiver given by the Dynkin quiver with an extra vertex.



The dimension vector is indicated in orange.

Nakajima quiver varieties

Put another way: for a dimension vector **d** we consider the representation.

$$E_{\mathbf{d}} = \oplus_{i\to j}\mathrm{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j}) \circlearrowright \prod G_{\mathbf{d}} = \mathrm{GL}(\mathbb{C}^{d_i})$$

The quotient here would be the moduli space of representations of the quiver. We want to take its hyperkähler analogue:

$$\tilde{\mathfrak{Q}}^{\lambda}_{\mathbf{d}} = \mu^{-1}(0) /\!\!/_{\chi} G_{\mathbf{d}} \subset T^*E_{\mathbf{d}} /\!\!/_{\chi} G_{\mathbf{d}}.$$

This has a natural resolution of singularities $\pi^{\lambda}_{\mathbf{d}} : \tilde{\mathfrak{Q}}^{\lambda}_{\mathbf{d}} \to \mathfrak{Q}^{\lambda}_{\mathbf{d}}$, where $\mathfrak{Q}^{\lambda}_{\mathbf{d}}$ the categorical quotient or moduli space of semi-simple preprojective representations. This makes $\tilde{\mathfrak{Q}}^{\lambda}_{\mathbf{d}}$ a symplectic resolution.

Nakajima quiver varieties

We let $\tilde{\mathfrak{Q}}^\lambda = \bigsqcup \tilde{\mathfrak{Q}}^\lambda_{\mathbf{d}}$, and $\mathfrak{Q}^\lambda = \bigcup \mathfrak{Q}^\lambda_{\mathbf{d}}$ (the inclusion is by adding the trivial representation).

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Unfortunately, the correspondence is a little more complicated than just sending the homology class of the component to the canonical basis vector. We live in an imperfect world.

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Recall that in my previous lecture, I defined a diagramatic algebra E^λ associated to $\mathfrak{g}$ and λ called the **quiver Hecke algebra**.

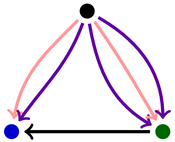
Theorem (W)

The derived category $D^b(\mathcal{O}_{\tilde{\mathfrak{Q}}^\lambda})$ carries an action of Khovanov and Lauda's 2-category categorifying $U_q(\mathfrak{g})$, and

$$D^b(\mathcal{O}_{\tilde{\mathfrak{Q}}^\lambda}) \cong D^b(E^\lambda\text{-mod}).$$

Nakajima quiver varieties

We can also get tensor products by incorporating a $\mathbb{C}^*$-action into the picture. If $\lambda = \lambda_1 + \cdots + \lambda_n$, we can partition our edges into groups corresponding to these weights, and act on the λ_i ones with weight i.



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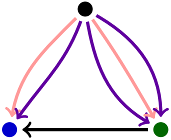

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The category $\mathcal{O}^{\underline{\boldsymbol{\lambda}}}$ for this $\mathbb{C}^*$ action on $\tilde{\mathfrak{Q}}^\lambda$ is a categorification of $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$.

Conjecture

$$D^b(\mathcal{O}^{\underline{\boldsymbol{\lambda}}}) \cong D^b(E^{\underline{\boldsymbol{\lambda}}}\text{-mod})$$

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For any sequence $\underline{\boldsymbol{\lambda}}$ of weights, we have a variety

$$\mathsf{Gr}_{\underline{\boldsymbol{\lambda}}} = \overline{G_{\lambda_1}} \times_{G[[t]]} \cdots \times_{G[[t]]} \overline{G_{\lambda_n}}/G[[t]] \qquad m_{\underline{\boldsymbol{\lambda}}} : \mathsf{Gr}_{\underline{\boldsymbol{\lambda}}} \to \overline{\mathsf{Gr}_\lambda}$$

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Conjecture

The shifted Yangian $Y_{\mu}(\mathfrak{g})$ is a deformation quantization of $K \cdot \mu(t)$. Category $\mathcal{O}$ for a quotient $Y_{\mu}^{\underline{\boldsymbol{\lambda}}}(\mathfrak{g})$ will be a block of the category $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}$ I defined in my talk yesterday.

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- If one takes a $\mathbb{C}^*$ action on $\mathfrak{Q}_\mu^\lambda$ for $\underline{\boldsymbol{\lambda}}$, and a partial resolution $\mathfrak{W}_\mu^{\underline{\boldsymbol{\lambda}}}$ of $\mathfrak{W}_\mu^\lambda$, components of the flow-in varieties (interpreted carefully) now are in bijection with the canonical basis of the tensor product $V_{\underline{\boldsymbol{\lambda}}} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_\ell}$.

Knot homology

So, if you believe me that the categories for symplectic dual manifolds are equivalent, this means that the knot homology construction I discussed yesterday should also have an affine Grassmannian interpretation.

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One can see a hint of where the braiding should come from: $\overline{\mathsf{Gr}_\lambda}$ is the special fiber of a family over the configuration space of ℓ-points in $\mathbb{C}$, where the general fiber is $\prod \overline{\mathsf{Gr}_{\lambda_i}}$ (coming from the Beilinson-Drinfeld Grassmannian) so the braiding functors are almost certainly related to monodromy in this family.

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The varieties $T^*GL_n/P_{^t\lambda}$ and $\tilde{\mathcal{S}}_\lambda$ are related by S-duality.

Thus, S-duality gives a general framework that includes this coincidence of knot invariants.

Future goals

- Find the true statements which lie behind all these conjectures.