

# Andrei Zelevinsky - Mutations for Quivers

Note Title

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with Potential

Ongoing Work with Harm Derksem & Jerzy Weyman

[Cluster algebras: motivations behind  
the scenes]

A quiver  $Q = (Q_0, Q_1, t, h)$

$Q_0$  = vertices (finite set) =  $\{1, \dots, n\}$

$Q_1$  = arrows (finite set)

$t, h: Q_1 \rightarrow Q_0$  tail & head:

$$t(a) \xrightarrow{a \in Q_1} h(a)$$

use as recording device for collections  
of linear maps: from where to where.

A  $Q$ -representation is a collection

( $M_i: i \in Q_0$ ,  $M(a): M(t(a)) \rightarrow M(h(a))$ )

fin dim  
vector spaces

linear map following  
arrow  $a \in Q_1$

Typical problem: classify all representations of a quiver

- e.g.  $\mathbb{Q}$ : classify metrics / conjugation  
Jordan blocks

$\mathbb{G}$ : Kronecker quiver -  
classify pairs of linear maps  
up to change of basis.

A Goal: classify indecomposable representations

e.g. single Jordan blocks in case  $\mathbb{Q}$

Classical tool: Reflection functors  
(Bernstein - Gelfand - Ponomarev '73)

- pick a vertex  $k$  which is a  
source or a sink (all outgoing or  
 all incoming)

$\mu_k$ : Reflection at  $k$  takes quiver  
 $\mathbb{Q}$  to quiver  $\overline{\mathbb{Q}}$ :

reverse all arrows at k

On representations :  $M \text{ Q-rep} \Rightarrow \bar{M} \text{ Q-rep}$

- preserve all vector spaces

$$\bar{M}_i = M_i \text{ for } i \neq k$$

at k : eg assume k is a sink

Let  $M_{in} = \bigoplus_{\substack{a \nearrow k \\ \text{all incoming} \\ \text{arrows}}} M_{t(a)}$  all incoming vector spaces (possibly with multiplicities)

$$M_{in} \xrightarrow{\alpha} M_k$$

$$\text{Set } \bar{M}_k = \text{Ker } \alpha$$

Reversed arrows  $\bar{M}(a^*) : \bar{M}_k \rightarrow M_{t(a)}$

- projection onto component  $M_{t(a)}$  of  $M_{in} \supset \bar{M}_k$ .

In case of a source : build  $M_{out} = \bigoplus_{\substack{t \rightarrow b \\ b \in S}} M_{h(b)}$

$$M_k \xrightarrow{\beta} M_{out}, \text{ take } \bar{M}_k = (\text{oker } \beta)^*$$

$\Rightarrow$  beautiful proof of Gabriel's theorem  
[cf. K. McGerty's GRASP lecture]

What happens when we write twice?

$\mu_k^2 : \text{sink} \rightarrow \text{source} \rightarrow \text{sink}$

$\bar{M}_k = \text{Im } \alpha : \text{only square}$   
to identity when  $\alpha$  is surjective,  
in general lose  $\text{Coker } \alpha$ .

Likewise for source lose  $\text{Ker } \beta$ .

Solution to this problem (Zelinsky w/ Marsh,  
Reineke) : create storage place  
for lost information!

Decorated quiver representations:

(create box near each vertex : add  
an extra vertex for each old vertex  
& no arrows)

$\rightarrow$  decorated representation is a pair  
 $(M, V)$   $M$  a  $Q$ -rep,  $V = (V_i)_{i \in Q_0}$

## Modified functors

$$\mu_k : (\mathcal{M}, \mathcal{V}) \longrightarrow (\bar{\mathcal{M}}, \bar{\mathcal{V}})$$

Don't change  $M_i, V_i$  if  $i \neq k$

$$\bar{M}_k = \ker \alpha \oplus V_k \quad \bar{V}_k = \text{coker } \alpha$$

$$\text{for source: } \bar{M}_k = \text{coker } \gamma_3 \oplus V_k, \quad \bar{V}_k = \text{Ker } \gamma_3.$$

These  $\mu_k$  do satisfy  $\mu_k^2 \approx \text{Id}$ , but  
not coherently lose functoriality  
(we've split vector spaces)

What to do with other vertices?

Suppose  $Q$  has no loops  & no  
oriented 2-cycles . Let  $k$  be any vertex,

$$\bar{Q} = \mu_k(Q) \quad \text{mutation of } Q$$

Step 1 Add some arrows: For each  $a \xrightarrow{k} b$ ,  
add  $a \xrightarrow{k} b$  "compose"  
 $[ba]$

Step 2 Reverse all arrows of  $\kappa$ :

replace each  $\xrightarrow{a \rightarrow k}$  by  $\xleftarrow{a^* \rightarrow k}$   $\xrightarrow{k \rightarrow b}$  by  $\xleftarrow{k^* \rightarrow b}$

Step 3 Rename oriented 2-cycles,  
one by one.

Check  $\mu_6^2 = \text{id}$  on quivers.

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Note usual reflection functors give  
bijections on indecomposable objects.

Here a problem arises!

Example  $a \xrightarrow{a^*} \rightsquigarrow \rightsquigarrow \xrightarrow{b^*} \rightsquigarrow \xrightarrow{b^*} [ba]$

Can't have bijection on indecomposables!

- must impose relations on oriented  
cycles we've created.

Above case: impose relation that all three  
compositions of two maps is zero:

$$a^* b^* = [ba] a^* = b^* [ba] = 0$$

$\Rightarrow$  set same number of indecomposables.

So need mutations on quivers with relations!

## Quivers with potentials (QP)

(related to superpotentials in physics...)

Completed path algebra of  $Q$ !

$R$  ground ring =  $K^{Q_0} = \text{Span } \{e_1, \dots, e_n\}$

orthogonal idempotents  $e_i e_j = \delta_{ij} e_i$

... "lazy path" starting at  $i$ .

$A = \text{linear span of all arrows}$

$R\langle\langle A\rangle\rangle = \{ \text{possibly infinite linear combinations } a_1 c_1 + \dots + a_m c_m \}$   
of paths  $c_1, c_2, \dots, c_d \xrightarrow{c_1} \xrightarrow{c_2} \dots \xrightarrow{c_d} c_m$   
multiplication = concatenation if defined  
or zero.  
[with adic topology]

Potentials:  $S \in R\langle\langle A \rangle\rangle_{\text{cyc}} = \text{Span of cyclic paths}$ : start & end at same point.

Consider up to cyclic

equivalence - as necklaces,  
can turn it around:

$$a_1 \dots a_d \sim a_2 \dots a_d a_1 .$$

i.e.  $S \sim S'$  if  $S - S' \in \text{closure of Span of turns}$ .

Potential  $S \rightsquigarrow$  2-sided ideal:

generalization of Jacobian ideal in  
singularity theory: generated by derivatives  
of potential.

Cyclic derivatives (Rota, Sagan, Stein '80):

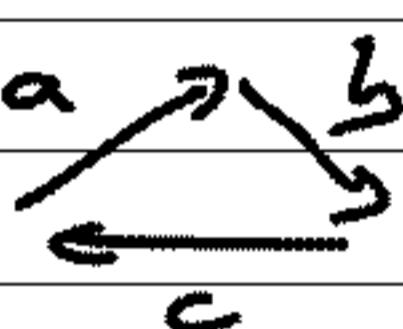
$$\xi \in A^* \quad \partial_\xi : R\langle\langle A \rangle\rangle_{\text{cyc}} \rightarrow R\langle\langle A \rangle\rangle$$

continuous linear  $\rightarrow$  enough to define  
on cyclic paths:

$$\partial_{\xi} (a_1 \dots a_d) = \sum_{k=1}^d \xi(a_k) a_1 \dots a_{k-1} a_{k+1} \dots a_d$$

Jacobian ideal  $J(S)$  = closure of the  
2-sided ideal generated by

$$2S = \{ \partial_{\xi} S, \xi CA^* \}$$

Example   $S = cba$

$$2S = \langle \partial_a S = cb, \partial_b S = ac, \partial_c S = ba \rangle$$

So these are the relations we want to impose.

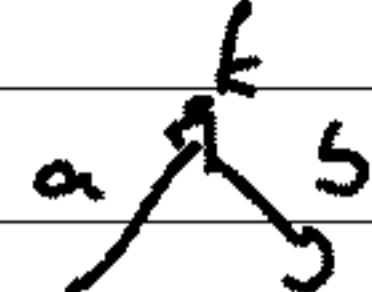
How do we mutate quivers with potentials?

$(Q, S)$  Quiver with potential  $\xrightarrow{\mu_n} (\bar{Q}, \bar{S})$

First two steps out of our three easy:

$(Q, S) \xrightarrow{\mu_n} (\tilde{Q}, \tilde{S})$

$$\tilde{S} := [S] + \Delta$$

$[S]$ : each tree a cyclic path  
goes through  $k$ , i.e. tree  
expressions  $\dots \text{ba} \dots$  for 

$\Rightarrow$  replace by  $\dots [ba] \dots$

$$\Delta = \sum_{\substack{a,b \\ a \geq b^*}} [ba] a^* b^*$$

$\xrightarrow{a^* \setminus b^*}$   
 $\xrightarrow{[ba]}$

- must consider  $(Q, S)$  up to right equivalence:  
if can identify path algebras as algebras,  
taking  $S$  to  $S'$ , ... not required  
to be homogeneous!