Classifying complex surfaces and symplectic 4-manifolds

Tim Perutz

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Symplectic 4-manifolds

Definition

A **symplectic 4-manifold** (X, ω) is an oriented, smooth, 4-dimensional manifold X equipped with a 2-form $\omega \in \Omega^2_X$ which is closed, i.e., $d\omega = 0$, and positive non-degenerate, i.e., $\omega \wedge \omega > 0$ everywhere on X.

- Our 4-manifolds will all be *compact* and *connected*.
- The conditions $d\omega = 0$ and $\omega \wedge \omega > 0$ can *individually* be understood through algebraic topology.
- *Jointly*, they are much more subtle. Existence of ω depends on the smooth structure, not just the homotopy-type.

Kähler surfaces

The 'classical' symplectic 4-manifolds are the *Kähler surfaces*.

Definition

A **Kähler surface** (X, ω) is a complex surface
(modeled on open sets of $\mathbb{C}^2$, holomorphic transition functions)
with a symplectic form ω of type $(1,1)$.
(locally $f(dz_1 \wedge d\bar{z}_2 + d\bar{z}_1 \wedge dz_2) + igdz_1 \wedge d\bar{z}_1 + ihdz_2 \wedge d\bar{z}_2$).

Examples:

- *Complex tori*: $\mathbb{C}^2$/lattice, $\omega = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$.
- *Projective algebraic surfaces* X: cut out from $\mathbb{C}P^N$ by homogeneous polynomials; ω_X = restriction of standard $PU(N+1)$-invariant 2-form on $\mathbb{C}P^N$.

Parity of the first Betti number

$b_1 = \dim H_1(X;\mathbb{R}) = \dim H^1(X;\mathbb{R})$.

Theorem

(i) *(Hodge, Weil, 1930s) A Kähler surface has even b_1.*

(ii) *(Classification-free proofs due to Buchdahl, Lamari, 1999) A complex surface with b_1 even admits Kähler forms.*

No such restriction exists for symplectic 4-manifolds. Let $T^2 = S^1 \times S^1$.

Example (Kodaira, Thurston)

There exists a symplectic 4-manifold X which is a symplectic T^2-bundle $T^2 \hookrightarrow X \to T^2$ with $b_1(X) = 3$.

By a *symplectic* surface-bundle I mean that one has a global symplectic form which is positive on the fibers.

Classification

- **Basic problem**: classify symplectic 4-manifolds. Various possible equivalence relations:
  - Symplectomorphism $f\colon X \to Y$ (i.e. diffeo with $f^*\omega_Y = \omega_X$)
  - Sometimes also allow symplectic deformations $\{\omega_t\}_{t\in[0,1]}$.
- **Inspiration**: the *Enriques–Kodaira classification* of Kähler surfaces. To what extent does it apply to symplectic 4-manifolds?
- **Tools**:
  - Cut-and-paste methods (Gompf, ...).
  - Pseudo-holomorphic curves (Gromov, ...); more powerful in symbiosis with
  - Seiberg–Witten gauge theory (Taubes, ...).
  - Lefschetz pencils (Donaldson, ...).

Blowing up

If X is a complex surface and $x \in X$, the *blow-up* of X at x is a complex surface $\widetilde{X}_x$ with a surjective holomorphic map $\pi \colon \widetilde{X}_x \to X$ (the *blow-down map*) such that

- $\pi \colon \pi^{-1}(X - \{x\}) \to X - \{x\}$ is bijective;
- $E := \pi^{-1}(x) \cong \mathbb{P} T_x X \cong \mathbb{C}P^1$.
- $N_E := T\widetilde{X}_x|_E / TE = \mathcal{O}(-1)$, i.e. $N_{E,\lambda} = \lambda$ for $\lambda \in \mathbb{P} T_x X$.

The infinitesimal geometry of X at x is seen macroscopically in $\widetilde{X}_x$.

Theorem (Castelnuovo)

If E is a smooth rational curve in a complex surface $\widetilde{X}$, and $\deg(N_E) = -1$, then there is a holomorphic map $\widetilde{X} \to X$ contracting E and realizing $\widetilde{X}$ as a blow-up.

Reduction to the minimal case

The construction of the blow-up $(\widetilde{X}_x, \widetilde{\omega})$ works in the symplectic category.

- There's now a parameter $\int_E \widetilde{\omega} > 0$.
- Blowing down is a natural symplectic surgery operation: if E is 2-sphere embedded in $\widetilde{X}$, with $\widetilde{\omega}|_E > 0$ and $\deg(N_E) = -1$, we can excise $\mathrm{nbhd}(E)$ and insert a 4-ball to construct X of which $\widetilde{X}$ is a blow-up.

Definition

A complex surface (resp. symplectic 4-manifold) is called *minimal* if it can't be blown down in the complex (resp. symplectic) sense.

Blowing down reduces b_2 by 1. Starting from a given X, we can blow it down a finite number of times to obtain a minimal X'. With certain easy exceptions, X' is determined by X.

The Enriques–Kodaira classification

Idea: distinguish *minimal* complex surfaces, with b_1 even, by the following measure:

$$\boxed{\text{How positive is the canonical line-bundle } \mathcal{K}_X?}$$

- $\mathcal{K}_X = \Lambda^2_{\mathbb{C}} T^*X$. Local sections $f\, dz_1 \wedge dz_2$.
- Positivity of line-bundles is a big topic in algebraic geometry, with many related notions of positivity. Here it means: *how fast does the dimension of the space of holomorphic sections $H^0(\mathcal{K}_X^{\otimes m})$ grow as $m \to \infty$?*
- For complex curves C, one has $\mathcal{K}_C = T^*C$ and for $m > 0$,
    - $\mathbb{C}P^1$: $\mathcal{K}$ negative, $H^0(\mathcal{K}^{\otimes m}) = 0$;
    - elliptic curves: $\mathcal{K}$ zero; $\dim H^0(\mathcal{K}^{\otimes m}) = 1$;
    - genus $g > 1$: $\mathcal{K}$ positive; for $m > 1$,

$$\dim H^0(\mathcal{K}^{\otimes m}) = (g-1)(2m-1) = O(m).$$

Dichotomy

The coarsest part of the Enriques–Kodaira classification is the Kodaira negative/non-negative dichotomy.

Definition

A minimal complex surface X is *Kodaira-negative* if $H^0(\mathcal{K}_X^{\otimes m}) = 0$ for all $m > 0$. Otherwise, it's Kodaira-non-negative.

There are other conditions which, not at all obviously, are equivalent to Kodaira negativity. They involve the the *canonical class* $K_X = c_1(\mathcal{K}_X) \in H^2(X; \mathbb{Z})$:

- There's a complex curve $C \subset X$ such that $\langle K_X, [C] \rangle < 0$.
- Either $K_X \cdot K_X < 0$ or, for each Kähler form ω, one has $K_X \cdot \omega < 0$.

Kodaira-negative surfaces

Theorem

Let X be a minimal complex surface, b_1 even.

(i) *(Castelnuovo): If X is Kodaira-negative and $K_X \cdot K_X > 0$ then X is either $\mathbb{C}P^2$ or a holomorphic $\mathbb{C}P^1$-bundle over $\mathbb{C}P^1$.*

(ii) *(Castelnuovo–de Franchis): If X is Kodaira-negative and $K_X \cdot K_X \leq 0$ then X is a holomorphic $\mathbb{C}P^1$-bundle over a curve.*

Symplectic Kodaira negativity

No holomorphic charts in the symplectic case, but we can make T^*X a complex vector bundle via an ω-compatible *almost complex structure* $J \in \mathrm{End}(TX)$, $J^2 = -I$. Define $K_X = c_1(\Lambda^2_{\mathbb{C}} T^*X)$.

Definition

A minimal symplectic 4-manifold (X, ω) is *Kodaira-negative* if $K_X \cdot \omega < 0$ or $K_X \cdot K_X < 0$.

Theorem (Aiko Liu, building on Gromov, McDuff, Taubes, Li–Liu)

Let (X, ω) be a minimal, Kodaira negative symplectic 4-manifold.

(i) *If $K_X \cdot K_X > 0$ then either X is symplectomorphic to $(\mathbb{C}P^2, r \cdot \omega_{std})$, or X is a symplectic S^2-bundle over S^2.*

(ii) *(Gompf's conjecture). If $K_X \cdot K_X \leq 0$ then X is a symplectic S^2-bundle over a surface.*

Kodaira-negative manifolds are S^2-bundles: strategy

- Pick a compatible almost complex structure $J \in \mathrm{End}(TX)$. Suppose we can find a J-holomorphic 2-sphere $C \subset X$ (meaning that $J(TC) = TC$) such that $\deg N_C = 1$ (like a line in $\mathbb{C}P^2$) or 0 (like a fiber).
- Say $\deg N_C = 0$. We have a 2-dimensional moduli space $\mathcal{M}$ of J-holomorphic spheres C' homologous to C, filling up an open subset $X' \subset X$ by Fredholm theory. If $\int_C \omega$ is minimal, X' is also closed by Gromov compactness, so $X' = X$.
- No two spheres $C', C'' \in \mathcal{M}$ intersect, because local intersection multiplicities of J-holomorphic curves are positive but $[C'] \cdot [C''] = \deg(N_C) = 0$.
- Mapping $x \in X$ to the unique $C \in \mathcal{M}$ on which it lies defines a symplectic bundle $S^2 \hookrightarrow X \to \mathcal{M}$.

... And then a miracle occurs

All this was posited on the existence of C.

- Existence of a suitable J-holomorphic sphere C in Kodaira-negative manifolds requires **magic**:

  $$\boxed{\text{the Seiberg–Witten equations.}}$$

- Specifically, use the SW wall-crossing formula to find cheap solutions, and Taubes's analysis to convert them to solutions localized on J-holomorphic curves.
- When $K_X^2 < 0$, the abelian instanton solutions to the SW equations play a role like that of the Albanese torus in the Castelnuovo–de Franchis theorem.

Kodaira dimension

X: minimal complex surface. We define its *Kodaira dimension* by

$$\operatorname{kod}(X) = \limsup \frac{\log P_n}{\log n}, \quad P_n := \dim H^0(\mathcal{K}_X^{\otimes n}).$$

Kodaira negative means $\operatorname{kod}(X) = -\infty$. When X is Kodaira-non-negative, the possibilities are

- $\operatorname{kod}(X) = 0 \Leftrightarrow \sup P_n = 1 \Leftrightarrow K_X$ is torsion in $H^2(X;\mathbb{Z})$; or
- $\operatorname{kod}(X) = 1$. This means that P_n grows linearly; or
- $\operatorname{kod}(X) = 2$. This means that P_n grows quadratically.

Kodaira dimension zero

Theorem (Enriques–Kodaira classification plus 'Torelli' theorems)

If X is a complex surface with K_X torsion then X belongs to one of the following types:

- *K3 surfaces: K_X trivial, $b_1(X) = 0$. One deformation class (that of a quartic surface in $\mathbb{C}P^3$).*
- *Enriques surfaces: K_X non-trivial, $2K_X$ trivial, $b_1(X) = 0$. One deformation class.*
- *Complex tori: $\mathbb{C}^2/(\text{lattice})$. One deformation class.*
- *Bi-elliptic and Kodaira surfaces: A non-trivial holomorphic fibre bundle over an elliptic curve, whose typical fiber is a genus 1 curve. Several deformation classes.*

Symplectic Kodaira dimension 0

Conjecture

If X is symplectic and K_X torsion then X is diffeomorphic to a K3 surface, an Enriques surface, or a symplectic T^2-bundle over T^2.

Theorem (S. Bauer, T.-J. Li, generalizing Morgan–Szabó)

If X is symplectic and K_X torsion then $b^+(X) \leq 3$. Consequently, $b_1(X) \leq 4$.

b^+ is the dimension of a maximal subspace of $H^2_{dR}(X)$ on which the quadratic form $Q(\alpha) = \int_X \alpha \wedge \alpha$ is positive-definite.
$b^+(K3) = 3$, $b^+(\text{Enriques}) = 1$.
The proofs use Seiberg–Witten theory—the quaternionic symmetry of the SW equations on a spin 4-manifold, the Bauer–Furuta homotopical SW theory, and the existence of a canonical SW solution when X is symplectic.

Symplectic Kodaira dimension 0

Conjecture

If X is symplectic and K_X torsion then X is diffeomorphic to a K3 surface, an Enriques surface, or a symplectic T^2-bundle over T^2.

Perhaps the conjecture can be demolished by a construction. Otherwise, it seems hard.
A weaker conjecture, perhaps more accessible (though SW theory?) is the following:

Conjecture

If K_X is torsion and $b_1(X) \leq 1$ then $2K_X = 0$.

Kodaira dimension 1

Kodaira dimension 1 (for a minimal complex surface) X means that P_m grows like $am+b$. In this case, one can associate a *canonical curve* C with X,

$$C = \mathrm{Proj}\left(\bigoplus_{m\geq 0} H^0(\mathcal{K}_X^{\otimes m})\right),$$

along with a map $X \to C$. Tweaking this construction one finds

Theorem

There's a holomorphic map $X \to B$ onto a curve B, such that K_X is trivial on the smooth fibers, which consequently have genus 1. Moreover, $X \to B$ has a section.

Deformation classes of such *elliptic surfaces* have been classified.

Symplectic Kodaira dimension 1

Definition

A minimal symplectic manifold (X, ω) has Kodaira dimension 1 if $K_X \cdot \omega > 0$ and $K_X \cdot K_X = 0$.

This agrees with the usual definition in the Kähler case. But the situation regarding classification is radically different.

Theorem (Gompf)

For any finite presentation $\langle g_1, \ldots, g_k \mid r_1, \ldots, r_\ell \rangle$ of a group G, one can construct a minimal symplectic 4-manifold X_P of Kodaira dimension 1 and an isomorphism $\pi_1(X_P) \cong G$.

Symplectic Kodaira dimension 1

Theorem (Gompf)

For any finite presentation $\langle g_1, \ldots, g_k \mid r_1, \ldots, r_\ell \rangle$ of a group G, one can construct a minimal symplectic 4-manifold X_P of Kodaira dimension 1 and an isomorphism $\pi_1(X_P) \cong G$.

- Distinguishing homotopy-types of 4-manifolds X which admit symplectic forms is therefore an algorithmically unsolvable problem. That's because an algorithm that could decide whether $\pi_1(X) = 1$ would solve the unsolvable halting problem will a Turing machine halt on a given program and input?.
- The construction uses a cut-and-paste method called *symplectic sum*. You start with a product $T^2 \times \Sigma_k$, and kill generators of π_1 by judicious symplectic sums along tori with a standard manifold $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$.

Kodaira dimension 2 (‘general type’)

- On minimal surfaces of general type, P_m grows quadratically.
- ‘Most’ surfaces have general type (e.g. hypersurface in $\mathbb{C}P^3$ cut out by a homogeneous polynomial of degree > 4).
- There’s no classification.

Theorem (Grothendieck–Bombieri)

Only finitely many deformation classes of general type surfaces share the same numbers $K \cdot K$ and e ($=$ topological Euler characteristic).

Theorem

Minimal surfaces of general type satisfy

(i) *(M. Noether)* $5K^2 - e + 36 \geq 6b_1$.

(ii) *(Bogomolov–Miyaoka–Yau)* $K^2 \leq 3e$.

Symplectic Kodaira dimension 2

On a minimal general type surface, for all Kähler forms ω, one has $K_X \cdot \omega > 0$ and $K_X \cdot K_X > 0$.

Definition

A minimal symplectic 4-manifold (X, ω) has Kodaira dimension 2 if $K_X \cdot \omega > 0$ and $K_X \cdot K_X > 0$.

- π_1 unconstrained.
- Few interesting invariants known, e.g. Gromov–Witten invariants seem to be rather uninformative.
- Noether's inequality is false.

Conjecture (Fintushel–Stern(?))

The BMY inequality $K \cdot K \leq 3e$ holds.

Summary

- Reduction to minimal manifolds works similarly in complex and symplectic categories.
- Kodaira-negative manifolds are S^2-bundles in both cases.
- Symplectic Kodaira-dimension 0 is on the cusp of our understanding. There *might* be a simple classification.
- Symplectic Kodaira dimension 1 is wild. (Better if we assume π_1 trivial?)
- Symplectic Kodaira dimension 2 is wild too, but we can ask for numerical constraints such as the conjectural symplectic BMY inequality.